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THE STEIN'S IDENTITY AND POISSON FUNCTIONALS

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Abstract. A new explicit construction of the stochastic derivative operator for Poisson polynomial functionals is introduced and some basic properties of stochastic derivative operator and Skorokhod integral are investigated.

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1. In the last quarter of the 20th century, in stochastic analysis and the measure theory the extended stochastic integral (the so-called Skorokhod integral) was constructed, where the independence of the integrand of the future is replaced by its smoothness in a certain sense (see, Skorokhod, 1975; Malliavin, 1978, 1997; Ocone, 1984; Nualart, Zakai, 1986; Nualart, Pardoux, 1988). It turned out (see, Gaveau, Trauber, 1982) that the operator of Skorokhod stochastic integration coincides with the conjugate operator of stochastic differentiation in the sense of Malliavin. As is known, the original aim of Malliavin's infinite-dimensional stochastic investigation was to study the density smoothness of a solution of a stochastic differential equation. The situation changed in 1991 when Karatas and Ocone showed how to apply in financial mathematics Ocone's theorem of stochastic integral representtation for the functional of diffusion processes. This theorem was subsequently called the Ocone-Haussmann-Clark formula.

G. Peccati (in 2009) explained how one can combine Stein's method with Malliavin calculus, in order to obtain explicit bounds in the normal and Gamma approximation of functionals of infinite-dimensional Gaussian fields. He showed that the Malliven operators are linked by several identities, all revolving around a fundamental result known at the integration by parts formula. This formula contains as a special case the "Stein's identity" $E[f'(\xi)-\xi f(\xi)]=0$ (where f is smooth bounded function and $\xi\cong N(0,1)$), which enters very naturally in the proof of closability of derivative operators. Malliavin's methods for jump processes (in particular, for Levy processes) were developed by Bichteler, Gravereaux, Jacod, Dermoune, Kree, Wu, Kabanov, Kaminski, Nualart, Vives, Picard, Di Nunno, Oksendal, Proske.

A further generalization of the Ocone-Clark formula belongs to Ma, Potter and Martin (1998) for the so-called normal martingale: if $F\in D^M_{2,1}$, then $F=EF+\int_0^T {}^p(D^M_t F)dM(t)$. As seen, this functional demands that the functional F would have the stochastic derivative. In that case, as different from the Wiener case, it is impossible to define the stochastic differentiation operator so that the Sobolev structure of the space $D^M_{2,1}$ could be obtained. Here the construction of the stochastic derivative is based on the expansion of a functional into a series of multiple stochastic integrals, whereas in the Wiener approach, in addition to this approach, use is also made of the Sobolev structure of a space.

2. Let $\sum_n := \{(t_1, ..., t_n) \in R_+^n : 0 < t_1 < \cdots < t_n\}$, and for a function f defined on $\sum_n$ define the multiple integral with respect to M as $I_n(f) = n! \int\limits_{\sum_n}\!\!\int f(t_1, ..., t_n) dM_{t_1} \cdots M_{t_n}$.

Definition 1. (see Definition 3.2 [3]). Let $\Re = \sigma(M_t; t \geq 0)$ be the σ-algebra generated by a normal martingale M. Let H_n be the n-th homogeneous chaos, $H_n = I_n(f)$, where f ranges over all $L_2(\sum_n)$. If $L_2(\Re, P) = \bigoplus\limits_{n=0}^{\infty} H_n$, then we say M possesses the chaos representation property (CRP).

Let $(\Omega, \Im, \{\Im_t\}_{t \geq 0}, P)$ be a filtered probability space satisfying the usual conditions. We assume that a normal martingale M with the CRP is given on it and that $\Im$ is generated by M. Thus, for any $F \in L_2(\Re, P)$ exists a sequence of functions $f_n \in L_s^2([0,1]^n)$, n=1,2,..., such that $F = \sum\limits_{n=0}^{\infty} I_n(f_n)$. $D_{2,1}^M := \{F = \sum\limits_{n=0}^{\infty} I_n(f_n) : \sum\limits_{n=1}^{\infty} nn! \|f\|_{L_2([0,1]^n)}^2 < \infty\}$.

Definition 2. (see [3]). The derivative operator is defined as a linear operator $D_{\cdot}^M$ from $D_{2,1}^M$ into $L^2([0,T] \times \Omega)$ by the relation:

$$D_t^M F := \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), t \in [0,1].$$

In the Wiener case except definition analogous to abovementioned there is a following equivalent definition: let w_t, $t \in [0,1]$ be a standard Wiener process defined on the canonical probability space $(\Omega, \Im, P)$, $\Im_t = \sigma(w_s, 0 \leq s \leq t)$. A smooth functional will be a random variable $F : \Omega \to R^1$ of the form $F = f(w_{t_1}, w_{t_2}, ..., w_{t_n})$, where $f \in C_b^\infty(R^n)$ and $t_1, t_2, ..., t_n \in [0,1]$. The derivative of F can be defined as (see [2]): $D_t^w F = \sum\limits_{n=1}^{\infty} \frac{\partial f}{\partial x^i}(w_{t_1}, w_{t_2}, ..., w_{t_n}) I_{[0,t_i]}(t)$.

3. Our aim is to study the basic properties of an explicit constructions of the stochastic derivative operator for compensated Poisson functionals, which was introduced by us in 2008 and which is not based on the chaos expansion of functionals, as well as in Ma, Protter and Martin's work. For brevity statements we consider here only a two-dimensional case so as it is enough for a clarification of our definition on an example which was considered earlier by Ma, Protter and Martin. It is necessary to notice that our definition is different from the definition of the Translation Operator introduced by Nualart and Vives in 1990. We will also investigate some properties of the Skorokhod integrals (in particular, we will give some examples for calculate of the Skorokhod integrals). In the end of work we will write down the Stein's identity in terms of the stochastic derivative.

Let $(\Omega, \Im, \{\Im_t\}_{t \in [0,T]}, P)$ be a filtered probability space satisfying the usual conditions. Let N_t be the standard Poisson process $(P(N_t = k) = t^k e^{-t}/k!, k = 0, 1, 2, ...)$ and $\Im_t$ is generated by $N (\Im_t = \Im_t^N), \Im = \Im_T$. Let M_t be the compensated Poisson process $(M_t = N_t - t)$. Let us denote $\nabla_x f(x) := f(x+1) - f(x)$; $\nabla_x f(M_T) := \nabla_x f(x)|_{x=M_T}$. For any function of two variables $g(\cdot, \cdot)$ introduce the designation: $\nabla^2 g(x,y) = g(x+1, y+1) - g(x,y)$.

It is not difficult to see that we can change in places an order of application of the operator $\nabla : \nabla_x[\nabla_y g(x,y)] = \nabla_y[\nabla_x g(x,y)]$ and the operator of the second order it is possible to calculate as follows: $\nabla^2 g(x,y) = \nabla_x[\nabla_y g(x,y)] + \nabla_x g(x,y) + \nabla_y g(x,y)$.

Using the following relations $M_s = \int\limits_0^T I_{[0,s]}(u)dM_s = I_1(I_{[0,s]}(\cdot))$,

$$M_{s-} = I_1(I_{[0,s)}(\cdot)) \quad \textit{and} \quad [M,M]_s = N_s = M_s + s,$$

by the Definition 2 we can respectively conclude that:

$$D_t^M M_s = D_t^M[I_1(I_{[0,s]}(\cdot))] = I_{[0,s]}(t),$$

$$D_t^M M_{s-} = I_{[0,s)}(t) \quad \textit{and} \quad D_t^M[M,M]_s = D_t^M M_s + D_t^M s = I_{[0,s]}(t).$$

Definition 3. (see Definition 4.1 [4]).

$$\overline{D}_t^M (M_s)^n := [\nabla_x(x^n)]|_{x=M_t} \cdot \overline{D}_t^M M_s := [\nabla_x(x^n)]|_{x=M_t} \cdot I_{[0,s]}(t);$$

for any polynomial function $P(x,y): \quad \overline{D}_t^M P(M_s, M_T) :=$

$$\nabla_y \nabla_x P(M_s, M_T) I_{[0,s]}(t) I_{[0,T]}(t) + \nabla_x P(M_s, M_T) I_{[0,s]}(t) + \nabla_y P(M_s, M_T) I_{[0,T]}(t).$$

Remark 1. As we see if $n = 1$, then the stochastic derivative for Wiener and Poisson processes formally are the same. The difference begins from the case $n = 2$. Indeed, if we take here $n = 2$, we obtain that $\overline{D}_t^M M_s^2 = \nabla_x x^2|_{x=M_s} \cdot \overline{D}_t^M M_s = (2M_s + 1)I_{[0,s]}(t)$, whereas in the Wiener process cases $D_t^w w_s^2 = \frac{\partial}{\partial x} x^2|_{x=w_s} \cdot D_t^w w_s = 2w_s I_{[0,s]}(t)$. This fact can be explained as follows: in the Wiener case in the definition of stochastic derivative the main component is an usual (classical) derivative, whereas in the Poisson case the main component is the operator ∇ and if $n = 1$ we have $x' = \nabla x = 1$, while if $n = 2$, then $2x = (x^2)' \neq \nabla x^2 = 2x + 1$.

Theorem 1. *For two-dimensional Poisson polynomial functional of the second degree the above-given two definitions of stochastic derivatives (Definition 3.2 from [3] and Definition 3.1) are equivalent:* $\overline{D}_t^M P_2(M_S, M_T) = D_t^M P_2(M_S, M_T)$.

Proof. At first we will prove the equivalence of definitions for double stochastic integrals, i. e. if $F = I_2(f_2)$ for some $f_2 \in L_s^2([0,T]^2)$, then F have the stochastic derivative, $\overline{D}_t^M F = 2I_1(f_2(\cdot,t)) = D_t^M F$ and $||\overline{D}_t^M F||^2_{L_2([0,T]\times\Omega)} = 2 \cdot 2! \cdot ||f_2||^2_{L_2([0,T]^2)}$.

Step 1. Suppose that f_2 is a symmetric function of the form $f_2(t_1,t_2) = aI_{A_1\times A_2} + aI_{A_2\times A_1}$, where $A_1,\ A_2 \subset [0,T]$, $A_1 \bigcap A_2 = \emptyset$. The set of such symmetric function we denote by E_2. For such f_2 we have

$$I_2(f_2) = 2a \int_0^T I_{A_1}(s)dM_s \int_0^T I_{A_2}(s)dM_s = 2aM(A_1)M(A_2).$$

Therefore, due to the Definition 3, one can easily verify that:

$$\overline{D}_t^M I_2(f_2) = 2a[I_{A_1}(t)I_{A_2}(t) + I_{A_1}(t)M(A_2) + I_{A_2}(t)M(A_1)] = 2I_1(f_2(\cdot,t)). \quad (1)$$

Moreover, it is not difficult to see that:

$$||\overline{D}_t^M F||^2_{L_2([0,T]\times\Omega)} = 2\cdot 2!\cdot \int_0^T |f_2(\cdot,t)||^2_{L_2([0,T])}dt = 2\cdot 2!\cdot ||f_2||^2_{L_2([0,T]^2)}. \qquad (2)$$

Step 2. If $F = I_2(f_2)$ for some $f_2 \in L_s^2([0,T]^2)$, then F can be approximated in the $L_2(\Omega)$-norm by a sequence of multiple integrals $I_2(f_2^n)$ of elements $f_2^n \in E_2$ as $n \to \infty$. By the relations (1) and (2) applied to f_2^n we deduce that the sequence of derivatives $\overline{D}_\cdot^M f_2^n$ converge in $L_2([0,T]\times\Omega)$, which completes the proof of the theorem for $F = I_2(f_2)$.

Step 3. Further it is obvious that if $A_1 = [0,S]$ and $A_2 = [0,T]$, then $I_2(f_2) = 2aM_SM_T$. Hence, using the Theorem 4.2 [4], it is not difficult to finish the proof of theorem.

Proposition 1. $D_tP(M_S,M_T) = [P(M_S+1,M_T+1) - P(M_S,M_T+1)]I_{[0,S]}(t) + +[P(M_S,M_T+1) - P(M_S,M_T)]I_{[0,T]}(t) = \nabla_x P(M_S,M_T+1)D_tM_S + \nabla_y P(M_S,M_T+1)D_tM_T$.

Proof. The proof directly follows from Definition 3 after simple transformations.

Now we will formulate a rule of differentiation for multiplication which show that in Poisson case, in difference from Wiener case, the operator of stochastic differentiation doesn't satisfy the property of a usual derivative.

Proposition 2. *For any polynomial functions $F(x,y)$ and $G(x,y)$ we have*

$$D_t[F(M_S,M_T)G(M_S,M_T)] = G(M_S,M_T)D_tF(M_S,M_T)$$

$$+F(M_S,M_T)D_tG(M_S,M_T) + D_tF(M_S,M_T)D_tG(M_S,M_T).$$

Proof. Due to the Definition 3 it is not difficult to see that

$$D_t[F(M_S,M_T)G(M_S,M_T)] = [F(M_S+1,M_T+1)G(M_S+1,M_T+1) - F(M_S,M_T+1)$$

$$\times G(M_S,M_T+1)]I_{[0,S]}(t) + [F(M_S,M_T+1)G(M_S,M_T+1) - F(M_S,M_T)G(M_S,M_T)]I_{[0,T]}(t)$$

$$= G(M_S,M_T)D_tF(M_S,M_T) + F(M_S,M_T)D_tG(M_S,M_T) + D_tF(M_S,M_T)D_tG(M_S,M_T).$$

Remark 2. In the one-dimensional case it easy to see that for any polynomial functions $F(x)$ and $G(x)$ we have

$$D_t[F(M_T)\cdot G(M_T)] = F(M_T+1)\cdot D_tG(M_T) + D_tF(M_T)\cdot G(M_T).$$

Remark 3. In the Wiener case the indicator function of A belongs to $D_{2,1}^w$ if and only if $P(A)$ is equal to zero or one (see, Sekiguchi, Shiota 1985). Consider now the Poisson case. According to the chain rule (Proposition 2) we can write $DI_A = D(I_AI_A) = 2I_ADI_A + (DI_A)^2$; $DI_A(1 - 2I_A - DI_A) = 0$. Hence, $DI_A = 0$ (and due to the Proposition 4.2 [4] $I_A = P(A)$, i.e. $P(A) = 0$ or 1 as in the Wiener case) or $DI_A = 1 - 2I_A$.

The next theorem,which follows from the Proposition 2, shows to us how it is possible to bring or take out the random variable respectively in or from the Skorokhod integral.

Theorem 2. *Let u_t is Skorokhod integrable and $F(x,y)$ is a polynomial function. Then $F(M_S, M_T)u_t$ is Skorokhod integrable and we have*

$$\int_0^T F(M_s, M_T)u_t dM_t = F(M_s, M_T)\int_0^T u_t dM_t$$

$$-\int_0^T u_t D_t[F(M_s, M_T)]dM_t - \int_0^T u_t D_t[F(M_s, M_T)]dt, \quad (P-a.s.).$$

Proposition 3. *For the Skorokhod integral the the following relation is valid:*

$$\int_0^T M_T^n dM_t = (M_T+T)(M_T-1)^n - M_T^n T \equiv N_T(M_T-1)^n - M_T^n T, \quad (n \geq 0). \quad (3)$$

Proof. For the proof we will use the principle of mathematical induction. The case $n=0$ is trivial. If $n=1$, using the Theorem 2, we easily see that:

$$\int_0^T M_T dM_t = M_T^2 - M_T - T = (M_T+T)(M_T-1) - M_T T.$$

Suppose now that the relation (3) is true for $n-1$ and verify that (3) is fulfilled. Due to the Theorem 2, using the induction assumption, one can easily verify that

$$\int_0^T M_T^n dM_t = M_T \int_0^T M_T^{n-1} dM_t - \int_0^T M_T^{n-1} I_{[0,T](t)} dM_t - \int_0^T M_T^{n-1} I_{[0,T](t)} dt$$

$$= M_T[(M_T+T)(M_T-1)^{n-1} - M_T^{n-1}T] - [(M_T+T)(M_T-1)^{n-1} - M_T^{n-1}T] - M_T^{n-1}T$$
$$= (M_T+T)(M_T-1)^{n-1}M_T - M_T^n T - (M_T+T)(M_T-1)^{n-1} + M_T^{n-1}T - M_T^{n-1}T$$
$$= (M_T+T)(M_T-1)^n - M_T^n T = N_T(M_T-1)^n - M_T^n T.$$

Corollary 1. For any polynomial function $P_n(x) = \sum_{i=0}^{n} a_i x^i$ the following relation is valid: $\int_0^T P_n(M_T)dM_t = (M_T+T)P_n(M_T-1) - TP_n(M_T) = M_T P_n(M_T-1) - TD_t[P_n(M_T-1)]$.

Proposition 4. *a)* $\int\limits_0^t M_t M_T^n dM_s = (M_T-1)^n(M_t^2 - M_t - t) - tM_t \sum\limits_{i=0}^{n-1}(M_T - 1)^i M_T^{n-1-i}$;

b) $\int\limits_0^t M_{t-}^n I_{[0,t)}(s)dM_s = \int\limits_{[0,t)} M_{t-}^n dM_s = (M_{t-}+t)(M_{t-}-1)^n - tM_{t-}^n$;

c) $\int\limits_0^t M_{t-}M_T dM_s = (M_T-1)(M_{t-}M_t - M_{t-} - t) - tM_{t-}$;

d) $\int\limits_0^t M_{t-}M_{t-}^n I_{[0,t]}(s)dM_s = (M_T-1)^n(M_{t-}^2 - M_{t-} - t) - tM_{t-}\sum\limits_{i=0}^{n-1}(M_T-1)^i M_T^{n-1-i}$;

e) $\int\limits_0^t M_{s-}(M_t - M_{s-})dM_s = (M_t-1)\int\limits_0^t M_{s-}dM_s - \int\limits_0^t M_{s-}ds - \int\limits_0^t M_{s-}^2 dM_s$.

Remark 4. It is not difficult to see that the "Stein's identity" for the Poisson random variable $N_T = M_T + T \cong Po(T)$ one can rewrite as follows:

$$E[T\nabla f(N_T) - (N_T - T)f(N_T)] = 0.$$

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