Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 24, 2010

ON THE ACCURACY OF SOLUTION APPROXIMATION WITH RESPECT TO A SPATIAL VARIABLE FOR ONE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION

Peradze J., Dzagania B., Papukashvili G.

Abstract. An initial boundary value problem for a wave equation is considered. To obtain an approximate solution with respect to a spatial variable the Galerkin method is used and its error is estimated.

Keywords and phrases: Nonlinear beam equation, Galerkin method, method error.

AMS subject classification: 65M60, 65N15.

1. Statement of the problem. Let us consider the initial boundary value problem

$$
\begin{gathered}
u_{tt}(x,t)+\delta u_t(x,t)+\gamma u_{xxxxt}(x,t)+\alpha u_{xxxx}(x,t)\\
-\left(\beta+\rho\int_0^L u_x^2(x,t)\,dx\right)u_{xx}(x,t)\\
-\sigma\left(\int_0^L u_x(x,t)u_{xt}(x,t)\,dx\right)u_{xx}(x,t)=0,\\
0<x<L,\quad 0<t\le T,
\end{gathered}
\tag{1}
$$

$$
\begin{gathered}
u(x,0)=u^0(x),\quad u_t(x,0)=u^1(x),\\
u(0,t)=u(L,t)=0,\quad u_{xx}(0,t)=u_{xx}(L,t)=0,
\end{gathered}
\tag{2}
$$

where α, γ, ρ, σ, β and δ are the given constants among which the first four are positive numbers, while $u^0(x)$ and $u^1(x)$ are the given functions.

The equation (1) obtained by J. Ball [1] using the Timoshenko [2] theory describes the vibration of a beam. The problem of construction of an approximate solution for this equation is dealt with in [3]-[5].

2. Galerkin method. We write an approximate solution of the problem (1), (2) in the form $u_n(x,t)=\sum\limits_{i=1}^{n}u_{ni}(t)\sin\dfrac{i\pi x}{L}$, where the coefficients $u_{ni}(t)$ will be found by the Galerkin method from the system of equations

$$
\begin{gathered}
u''_{ni}(t)+\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)u'_{ni}(t)+\left[\alpha\left(\frac{i\pi}{L}\right)^4+\left(\frac{i\pi}{L}\right)^2\left(\beta+\rho\,\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2u_{nj}^2(t)\right.\right.\\
\left.\left.+\sigma\,\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2u_{nj}(t)u'_{nj}(t)\right)\right]u_{ni}(t)=0,\\
i=1,2,\ldots,n,\quad 0<t\le T,
\end{gathered}
\tag{3}
$$

with the initial conditions

$$u_{ni}(0) = a_i^0, \quad u'_{ni}(0) = a_i^1, \quad i = 1, 2, \ldots, n. \tag{4}$$

Here a_i^0 and a_i^1 are the coefficients from the representation of the functions $u^0(x)$ and $u^1(x)$ as $u^p(x) = \sum\limits_{i=1}^{\infty} a_i^p \sin \dfrac{i\pi x}{L}$, $p = 0, 1$.

3. Method error. If

$$u^p(x) \in L^2(0, L), \quad p = 0, 1, \tag{5}$$

then there exists a generalized solution of the problem (1), (2) that is a function $u(x,t)$ representable as a series $\sum\limits_{i=1}^{\infty} u_i(t) \sin \dfrac{i\pi x}{L}$, the coefficients of which satisfy the system of equations

$$\begin{aligned} u''_i(t) + \left(\delta + \gamma \left(\frac{i\pi}{L}\right)^4\right) u'_i(t) + \Bigg[\alpha \left(\frac{i\pi}{L}\right)^4 + \left(\frac{i\pi}{L}\right)^2 \Bigg(\beta + \rho\, \frac{L}{2} \sum_{j=1}^{\infty} \left(\frac{j\pi}{L}\right)^2 u_j^2(t) \\ +\sigma\, \frac{L}{2} \sum_{j=1}^{\infty} \left(\frac{j\pi}{L}\right)^2 u_j(t) u'_j(t)\Bigg)\Bigg] u_i(t) = 0, \\ i = 1, 2, \ldots, \quad 0 < t \le T, \end{aligned} \tag{6}$$

with the initial conditions

$$u_i(0) = a_i^0, \quad u'_i(0) = a_i^1, \quad i = 1, 2, \ldots\ . \tag{7}$$

Denote $\Delta u_{ni}(t) = u_{ni}(t) - u_i(t)$ and assume that under the method error we will understand the function $\Delta u_n(x,t) = \sum\limits_{i=1}^{n} \Delta u_{ni}(t) \sin \dfrac{i\pi x}{L}$ the $L^2(0,L)$-norm of which we want to estimate.

Subtract (6) from (3) and, having multiplied the resulting equality by $2\Delta u'_{ni}(t)$, sum it over $i = 1, 2, \ldots, n$. We obtain

$$\begin{aligned} &\frac{d}{dt} \sum_{i=1}^{n} \Delta u'^2_{ni}(t) + 2 \sum_{i=1}^{n} \left(\delta + \gamma \left(\frac{i\pi}{L}\right)^4\right) \Delta u'^2_{ni}(t) \\ &\qquad + \frac{d}{dt} \sum_{i=1}^{n} \left(\alpha \left(\frac{i\pi}{L}\right)^4 + \beta \left(\frac{i\pi}{L}\right)^2\right) \Delta u^2_{ni}(t) \\ &+ \rho\, \frac{L}{2} \Bigg[\sum_{j=1}^{n} \left(\frac{j\pi}{L}\right)^2 (u_{nj}(t) + u_j(t)) \Delta u_{nj}(t) \sum_{i=1}^{n} \left(\frac{i\pi}{L}\right)^2 (u_{ni}(t) + u_i(t)) \Delta u'_{ni}(t) \\ &\qquad + \frac{L}{2} \sum_{j=1}^{n} \left(\frac{j\pi}{L}\right)^2 (u^2_{nj}(t) + u^2_j(t))\, \frac{d}{dt} \sum_{i=1}^{n} \left(\frac{i\pi}{L}\right)^2 \Delta u^2_{ni}(t) \Bigg] \\ &+ \frac{\sigma L}{4} \Bigg\{ \sum_{j=1}^{n} \left(\frac{j\pi}{L}\right)^2 \Big[(u'_{nj}(t) + u'_j(t)) \Delta u_{nj}(t) + (u_{nj}(t) + u_j(t)) \Delta u'_{nj}(t) \Big] \end{aligned}$$

$$\times \sum_{i=1}^{n} \left(\frac{i\pi}{L}\right)^2 (u_{ni}(t)+u_i(t))\Delta u'_{ni}(t)$$
$$+ \sum_{j=1}^{n} \left(\frac{j\pi}{L}\right)^2 (u_{nj}(t)u'_{nj}(t)+u_j(t)u'_j(t)) \sum_{i=1}^{n} \left(\frac{i\pi}{L}\right)^2 \frac{d}{dt}\Delta u^2_{ni}(t)\Big\}$$
$$= L\Big(\rho \sum_{j=n+1}^{\infty} \left(\frac{j\pi}{L}\right)^2 u_j^2(t) + \sigma \sum_{j=n+1}^{\infty} \left(\frac{j\pi}{L}\right)^2 u_j(t)u'_j(t)\Big) \sum_{i=1}^{n} \left(\frac{i\pi}{L}\right)^2 \Delta u'_{ni}(t)u_i(t). \quad (8)$$

Subtracting (7) from (4) for $i = 1, 2, \dots, n$, we have

$$\Delta u_{ni}(0) = 0, \quad \Delta u'_{ni}(0) = 0, \quad i = 1, 2, \dots, n. \quad (9)$$

Our further consideration will be restricted to a more difficult case when β and δ are negative numbers. For three other combinations of these numbers we come to a result analogous to the one given at the end of this paper.

Multiply the equation (3) by $2u'_{ni}(t)$ and the equation (6) by $2u'_i(t)$. Then sum the obtained equalities over $i = 1, 2, \dots, n$ in the first case, and over $i = 1, 2, \dots$ in the second case. As a result, after some transformations, we make the following conclusion.

Lemma 1. *The estimates*

$$\Phi_n(t) \le c_{0n}, \quad \Phi(t) \le c_0 \quad (10)$$

are valid, where

$$\Phi_n(t) = \sum_{i=1}^{n} u'^2_{ni}(t) + \alpha \sum_{i=1}^{n} \left(\frac{i\pi}{L}\right)^4 u^2_{ni}(t) + \rho \frac{L}{4}\Big(\sum_{i=1}^{n} \left(\frac{i\pi}{L}\right)^2 u^2_{ni}(t)\Big)^2,$$
$$\Phi(t) = \sum_{i=1}^{\infty} u'^2_{i}(t) + \alpha \sum_{i=1}^{\infty} \left(\frac{i\pi}{L}\right)^4 u^2_{i}(t) + \rho \frac{L}{4}\Big(\sum_{i=1}^{\infty} \left(\frac{i\pi}{L}\right)^2 u^2_{i}(t)\Big)^2,$$
$$c_{0n} = \Phi_n(0)e^a \le \Phi(0)e^a = c_0,$$
$$a = \left(-\delta + \left(\delta^2 + \frac{\beta^2}{a}\right)^{\frac{1}{2}}\right) T.$$

Now multiply the equation (6) by $2u'_i(t)$ and sum the obtained equality over $i = n+1, n+2, \dots$. After a few transformations we see that the following statement is true.

Lemma 2. *The inequality*

$$F_n(t) \le F_n(0)e^b \quad (11)$$

is fulfilled, where

$$F_n(t) = \sum_{i=n+1}^{\infty} u'^2_{i}(t) + \alpha \sum_{i=n+1}^{\infty} \left(\frac{i\pi}{L}\right)^4 u^2_{i}(t) + \frac{\rho L}{4}\Big(\sum_{i=n+1}^{\infty} \left(\frac{i\pi}{L}\right)^2 u^2_{i}(t)\Big)^2,$$
$$b = \left(-\delta + \left(\delta^2 + \frac{\omega^2}{\alpha}\right)^{\frac{1}{2}}\right) T, \quad \omega = -\beta + c_0 \frac{L}{2\sqrt{\alpha}}\left(\frac{\sigma}{2} + \frac{\rho}{\sqrt{\alpha}}\left(\frac{L}{\pi}\right)^2\right).$$

Let us introduce the notation

$$\nu = \frac{1}{4}\, c_0 e^{2b} \left(\frac{L}{2\alpha}\right)^3 T \left(\rho \left(\frac{L}{\pi}\right)^2 + \left(\rho^2 \left(\frac{L}{\pi}\right)^4 + \alpha\sigma^2\right)^{\frac{1}{2}}\right)^2,$$

$$\nabla = -2\beta + (c_{0n} + c_0)\, \frac{L}{2\sqrt{\alpha}} \left(\sigma + 3\, \frac{\rho}{\sqrt{\alpha}} \left(\frac{L}{\pi}\right)^2\right)$$

and use (10) and (11) in (8) and (9). Thus we obtain the upper bound for $\sum\limits_{i=1}^{n} \Delta u_{ni}'^2(t) + \sum\limits_{i=1}^{n} \left(\frac{i\pi}{L}\right)^4 \Delta u_{ni}^2(t)$. Using it we come to the following result.

Theorem. *If the requirement* (5) *is fulfilled for the functions* $u^p(x)$, $p = 0, 1$, *and the above-mentioned conditions are fulfilled for the constants* α, γ, ρ, σ, β *and* δ, *then for the Galerkin method error the estimate*

$$\left\| \frac{\partial}{\partial t}\, \Delta u_n(x,t) \right\|^2_{L^2(0,L)} + \alpha \left\| \frac{\partial^2}{\partial x^2}\, \Delta u_n(x,t) \right\|^2_{L^2(0,L)} \le c \left(\sum_{i=n+1}^{\infty} a_i^{1\,2} + \sum_{i=n+1}^{\infty} \left(\frac{i\pi}{L}\right)^4 a_i^{0\,2} \right)^2$$

holds, where

$$c = e^{\frac{1}{2} T (\nu - 2\delta + ((\nu - 2\delta)^2 + \frac{\nabla^2}{\alpha})^{\frac{1}{2}})}.$$

Acknowledgement. The first author J. Peradze acknowledgements a support under the grant 7353 of Estonia Science Foundation.

R E F E R E N C E S

1. Ball J.M. Stability theory for an extensible beam. *J. Differ. Equations*, **14** (1973), 399-418.

2. Timoshenko S., Yong D., Weaver W. Vibration Problems in Engineering. *John Willey, New York*, 1967.

3. Choo S.M., Chung S.K. L^2-error estimate for the strongly damped extensible beam equations. *Appl. Math. Lett.* **11**, 6 (1998), 101-107.

4. Choo S.M., Chung S.K. Finite difference approximate solutions for the strongly damped extensible beam equations. *Appl. Math. Comput.* **112**, 1 (2000), 11-32.

5. Choo S.M., Chung S.K., Kannan R. Finite element Galerkin solutions for the strongly damped extensible beam equations. *Korean J. Comp. Appl. Math.*, **7** (2001), 1-20.

Received 11.06.2010; revised 11.10.2010; accepted 18.11.2010.

Authors' addresses:

J. Peradze
Iv. Javakhishvili Tbilisi State University
1/3, Chavchavadze Av., Tbilisi 0128
Georgia
E-mail: j_peradze@yahoo.com

B. dzagania
Iv. Javakhishvili Tbilisi State University
1/3, I. Chavchavadze Av., Tbilisi 0128
Georgia
E-mail: b_dzagania@yahoo.com

G. Papukashvili
Ilia State University
3/5, K. Cholokashvili St., Tbilisi 0162
Georgia
E-mail: papukashvili@yahoo.com