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ABOUT OF ONE BOUNDARY PROBLEM SOLUTION OF ANTIPLANE ELASTICITY THEORY BY INTEGRAL EQUATION METHODS

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Abstract. Antiplane problems of the theory of elasticity by using the theory of analytical functions are presented. These problems lead to a system of singular integral equations with immovable singularity with the respected to leap of the tangent stress. The problems of behavior of solutions at the boundary are studied. A singular integral equation containing an immovable singularity is solved by collocation method. It is shown that the system of the corresponding algebraic equations is solvable for sufficiently big number of the integral division. Experimental convergence of approximate solutions to the exact one is detected.

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AMS subject classification: 65M70, 34B05, 31A10, 45E05.

Natural phenomena, such as earthquakes and floods, may cause occurrence of cracks in the pipes of great diameter, and in gas-and petrol tanks. Increase of the number and dimensions of cracks, from its part, causes to increase the volume of leakage that will result in great echocatastrophe. The study of boundary value problems for the composite bodies weakened by cracks has a great practical significance (see [1]-[7]). Equations of the antiplane elasticity theory for composite bodies weakened by cracks can be used as an initial approximation of the mathematical model of problems provided by the project Grant $\#GNSF09-614\_5-210$. More interesting cases when cracks intersect an interface or penetrate the boundary at any angle will be investigated [8]-[10].

Let elastic Ω body occups complex variable plane $z=x+iy$ which is cut on the line $L=[-1;+1]$. The plane consists of two orthotropic homogeneous semiplanes

$$\Omega_1=\{z: Rez\geq 0, x\notin L_1=[0;1]\},$$

$$\Omega_2=\{z: Rez\preceq 0, x\notin L_2=[-1;0]\},$$

which are welded on the axis y. Define by index $k, k=1,2$ values and functions connected with Ω_k.

The problem is to find the function $W_k(x,y)$, which satisfies differential equation:

$$\frac{\partial^2 W_k(x,y)}{\partial x^2}+\lambda_k^2\frac{\partial^2 W_k(x,y)}{\partial y^2}=0, \quad (x,y)\in\Omega_k \tag{1}$$

and boundary conditions:

a) on the boundary of the crack tangent stresses are given:

$$b_{44}^{(k)}\frac{\partial W_k(x,\pm 0)}{\partial y}=q_k^{(\pm)}(x),\quad x\in L_k; \tag{2}$$

b) on the axis y the condition of continuity is fulfilled:

$$W_1(0;y)=W_2(0;y),\quad y\in(-\infty;\infty),\quad y\neq 0, \tag{3}$$

$$b_{55}^{(1)}\frac{\partial W_1(0;y)}{\partial x}=b_{55}^{(2)}\frac{\partial W_2(0;y)}{\partial x}, \tag{4}$$

where $\lambda_k^2=\dfrac{b_{44}^{(k)}}{b_{55}^{(k)}}$, $b_{44}^{(k)}$, $b_{55}^{(k)}$ are elastic constants, $q_k(x)$ is a function of Holder's class, $k=1,2$. In particular, if we have isotropic case $b_{44}^{(k)}=b_{55}^{(k)}=\mu_k$, $\lambda_k=1$, where μ_k is module of displacement, $k=1,2$.

If we use affine transformation and introduce the symbols:

$$W_k(\xi,\lambda_k\eta)\equiv\widetilde{W}_k(\xi,\eta),\quad \frac{\lambda_k}{b_{44}^{(k)}}q_k(\xi)\equiv f_k(\xi),\quad \frac{b_{55}^{(2)}}{b_{55}^{(1)}}\equiv\gamma_2,\quad k=1,2,$$

we get $x=\xi$, $y=\lambda_k\eta$, $\lambda_k>0$, $k=1,2$ and (1)-(4) boundary problems will have forms:

$$\Delta\widetilde{W}_k(\xi,\eta)\equiv\frac{\partial^2\widetilde{W}_k(\xi,\eta)}{\partial\xi^2}+\frac{\partial^2\widetilde{W}_k(\xi,\eta)}{\partial\eta^2}=0\quad(\xi,\eta)\in\Omega_k,$$

$$\frac{\partial\widetilde{W}_k(\xi,\pm 0)}{\partial\eta}=f_k(\xi),\quad \xi\in L_k. \tag{2'}$$

$$\widetilde{W}_1(0,\eta)=\widetilde{W}_2(0,\eta),\quad \eta\in(-\infty;\infty),\quad \eta\neq 0, \tag{3'}$$

$$\frac{\partial\widetilde{W}_1(0,\eta)}{\partial\xi}=\frac{\partial\widetilde{W}_2(0,\eta)}{\partial\xi}. \tag{4'}$$

Antiplane problems of the theory of elasticity by using the theory of analytical functions are presented in the paper. By using the functions of complex variable a unknown harmonic function $\widetilde{W}_k(\xi,\eta)$ will be presented in the following form:

$$\widetilde{W}_k(\xi,\eta)=\frac{1}{2}\left(\varphi_k(Z)+\overline{\varphi}_k(Z)\right),$$

where $\varphi_k(Z)=u_k(\xi,\eta)+iv_k(\xi,\eta),\quad Z=\xi+i\eta.$

Therefore

$$\widetilde{W}_k(\xi,\eta)=Re\varphi_k(Z).$$

These problems lead to a system of singular equations with immovable singularity with the respected to leap of the tangent stress. The problems of behavior of solutions at the boundary are studied. With using the theory of analytical functions (in particular, we use formulas of definition of piecewise holomorphic functions for given leap), also boundary value Sokhotski-Plemel formula of Cauchy type integral [11], from boundary conditions $(2') - (4')$ the system of singular integral equations with respect to leaps $\rho_k(\xi)$

$$\begin{aligned} &\int_0^1 \left(\frac{1}{t-\xi} - \frac{a_1}{t+\xi}\right)\rho_1(t)dt + b_1\int_{-1}^0 \frac{\rho_2(t)dt}{t-\xi} = 2\pi f_1(\xi), \;\; \xi \in (0,1), \\ &b_2\int_0^1 \frac{\rho_1(t)dt}{t-\xi} + \int_{-1}^0 \left(\frac{1}{t-\xi} - \frac{a_2}{t+\xi}\right)\rho_2(t)dt = 2\pi f_2(\xi), \;\; \xi \in (-1,0), \end{aligned} \tag{5}$$

where $\rho_k(x) = \Phi_k^+(x) - \Phi_k^-(x)$ real functions, $\Phi_k(Z) = \varphi_k'(z)$,

$$a_k = \frac{1-\gamma_k}{1+\gamma_k}, \quad b_k = \frac{2}{1+\gamma_k}, \quad \gamma_1 = 1/\gamma_2, \quad \rho_k(x) \in H^*, \quad k = 1, 2.$$

Explanation of behavior of solutions near the ends of the boundary presents a special interest. The solutions of the system (5) of the integral equations can be presented in the following way:

$$\rho_1(t) = \frac{\chi_1(t)}{t^{\alpha_1}(1-t)^{\beta_1}}, \quad \rho_2(t) = \frac{\chi_2(t)}{t^{\alpha_2}(1+t)^{\beta_1}},$$

where α_k, β_k are unknown constants $0 < \alpha_k$, $\beta_k < 1$, and $\chi_k(t)$ are functions, which belong to Holder's class, $k = 1, 2$. In the point $t = \pm 1$ we obtain correspondingly $\beta_1 = \beta_2 = \frac{1}{2}$. In the considered case there is no peculiarity in the point $t = 0$.

In a partial case when one half-plane has a rectilinear cut of finite length, which is perpendicular to the boundary , and one end of which is located on the boundary. We have one singular integral equation containing an immovable singularity. In a partial case, when a crack reaches the boundary of separation, we get that an order of peculiarity in the point depends on elastic constants of material and belongs to $0 < \alpha < 1$. Let, $\rho_2(x) \equiv 0$, $\rho_1(x) \neq 0$, then integral of the system (5) we have one integral equation:

$$\int_0^1 \left(\frac{1}{t-\xi} - \frac{a_1}{t+\xi}\right)\rho(t)dt = 2\pi f_1(\xi), \;\; \xi \in (0,1).$$

We get that an order of peculiarity in the point $t = 0$ depends on elastic constants of material and belongs to interval (0;1). $\alpha = 1 - \frac{1}{\pi} arccos\left(\frac{b_{55}^{(1)} - b_{55}^{(2)}}{b_{55}^{(1)} + b_{55}^{(2)}}\right) \in (0;1)$, $\beta = \frac{1}{2}$.
If $b_{55}^{(1)} = b_{55}^{(2)}$, then $\alpha = \frac{1}{2}$.

In the work we use the collocation method. We consider cases as regular intervals, so nonregular located knots in relation to the integrated variable. In the first case the integral is replaced with the quadrature formula of open type, and in the second case the quadrature formula of the higher accuracy. We take roots of polynoms Chebishev of the first sort as knots. For approximate solution of the above-mentioned problem we form the program. The program is examined with testing problem. Several numerical experiments gave the satisfactory results.

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