Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 24, 2010

# NUMERICAL RESOLUTION OF ONE NONLINEAR PARTIAL DIFFERENTIAL SYSTEM

Nikolishvili M.

**Abstract**. The one-dimensional analog of the nonlinear system of partial differential equations arising in process of vein formation of young leaves is considered. Finite difference schemes for initial-boundary value problems are constructed. Numerical experiments are done and respective graphical illustrations are presented.

**Keywords and phrases**: System of nonlinear partial differential equations, vein formation of young leaves, finite difference schemes.

**AMS subject classification**: 35Q80, 65N06, 65Y99.

In [1] system of two-dimensional nonlinear differential equations is proposed which is connected with process of vein formation in meristematic tissues of young leaves.

Note, that one-dimensional analogous of this system is studied in work [2]. Here the following boundary-value problem is considered:

$$\begin{gathered}
\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V\frac{\partial U}{\partial x}\right),\quad (x,t)\in(0,1)\times(0,T],\\
\frac{\partial V}{\partial t}=-V+g\left(V\frac{\partial U}{\partial x}\right),\quad (x,t)\in[0,1]\times(0,T],\\
U(0,t)=0,\quad V\frac{\partial U}{\partial x}\Big|_{x=1}=\psi,\quad t\in[0,T],\\
U(x,0)=U_0(x);\quad V(x,0)=V_0(x)\geq\delta_0,\quad x\in[0,1],
\end{gathered}\tag{1}$$

where $g$, $U_0$, $V_0$ are known sufficiently smooth functions, $g_0\leq g(\xi)\leq G_0$; $T$, $g_0$, $G_0$, $\delta_0$, $\psi$ are given positive constants.

Theorems of solvability, uniqueness and asymptotic behavior of solution of problem (1) are proved in [2]. Using following notation $W(x,t)=V\frac{\partial U}{\partial x}$ the authors reduced problem (1) to the equivalent form:

$$\begin{gathered}
\frac{\partial W}{\partial t}=V\frac{\partial^2 W}{\partial x^2}+\left[\frac{g(W)}{V}-1\right]W,\\
\frac{\partial V}{\partial t}=-V+g(W),\\
\frac{\partial W}{\partial x}\Big|_{x=0}=0,\quad W(1,t)=\psi,\\
W(x,0)=W_0(x)=V_0(x)\frac{dU_0(x)}{dx},\quad V(x,0)=V_0(x).
\end{gathered}\tag{2}$$

The solvability of solutions of the problem (2) are studied by following iterative procedure:

$$
\begin{gathered}
\frac{\partial W^n}{\partial t}=V^{n-1}\frac{\partial^2 W^n}{\partial x^2}+\left[\frac{g\left(W^{n-1}\right)}{V^{n-1}}-1\right]W^{n-1},\\
\frac{\partial V^n}{\partial t}=-V^n+g\left(W^{n-1}\right),\\
\left.\frac{\partial W^n}{\partial x}\right|_{x=0}=0,\quad W^n(1,t)=\psi,\\
W^n(x,0)=W_0(x),\quad V^n(x,0)=V_0(x),\\
n=1,2,\cdots.
\end{gathered}
\tag{3}
$$

Many scientific works are devoted to the investigation and numerical resolution of (1) type problem and its multi-dimensional analogs (see, for example, [2]-[7] and references therein)

The purpose of our note is to construct approximate solutions of the problems (1) and (2) by using finite difference schemes.

For problem (1) in [3] the following semi-discrete scheme is constructed and the convergence in the norm of the usual space $C_h$ is studied:

$$
\begin{gathered}
\frac{du}{dt}-\left(vu_{\bar{x}}\right)_x=0,\\
\frac{dv}{dt}=-v+g\left(vu_{\bar{x}}\right),\\
u(x,0)=U_0(x),\quad v(x,0)=V_0(x),\\
u(0,t)=0,\quad v_{M-1/2}u_{\bar{x},M}+\frac{h}{2}\left.\left(\frac{du}{dt}\right)\right|_M=\psi.
\end{gathered}
\tag{4}
$$

Here function $u$ is defined on the set $\bar{\omega}_h\times[0,T]$ with the grid $\bar{\omega}_h=\{x_i=ih,\ i=0,1,...,M;\ h=1/M\}$ and the function $v$ on the set $\omega_h^*\times[0,T]$ with the grid $\omega_h^*=\{x_i=(i-1/2)h,\ i=1,2,...M\}$ respectively.

Let us problem (1) has sufficiently smooth solution [2]. The following statement takes place.

**Theorem 1.** *The semi-discrete scheme (4) converges to the solution of problem (1) in the norm of the space $C_h$ with the rate $O\left(h^2\right)$.*

On the grids $\bar{\omega}_h\times\omega_\tau$ and $\omega_h^*\times\omega_\tau$, where $\omega_\tau=\{t_j=j\tau,\ j=0,1,...,N,\ \tau=T/N\}$, the fully-discrete finite difference scheme for problem (1) based on (3) is also constructed and investigated [3]. This scheme has the form:

$$
\begin{gathered}
u_{t,i}^j=\left(v_i^{j+1}u_{\bar{x},i}^{j+1}\right)_x,\\
v_{t,i-1/2}^j=-v_{i-1/2}^{j+1}+g\left(v_{i-1/2}^{j+1}u_{\bar{x},i}^{j+1}\right),\\
u_i^0=U_{0,i},\quad v_{i-1/2}^0=V_{0,i-1/2},\\
u_0^j=0,\quad v_{M-1/2}^j u_{\bar{x},M}^j+\frac{h}{2}u_{t,M}^j=\psi.
\end{gathered}
\tag{5}
$$

**Theorem 2.** *The finite difference scheme (5) converges to the solution of problem (1) in the norm of the space $C_h$ with the rate $O\left(\tau+h^2\right)$.*

The algorithm of solution of nonlinear system of equations (5) is based on Newton iterative procedure [8].

The finite difference scheme of rate $O(\tau + h^2)$ for problem (2) is also constructed. The numerical solution of this scheme is realized by using (3) type iterative process.

The results of numerical experiments by the algorithms based on (3) type iterative process are given on the Fig. 1 and by algorithm based on (5) are shown on Fig. 2.

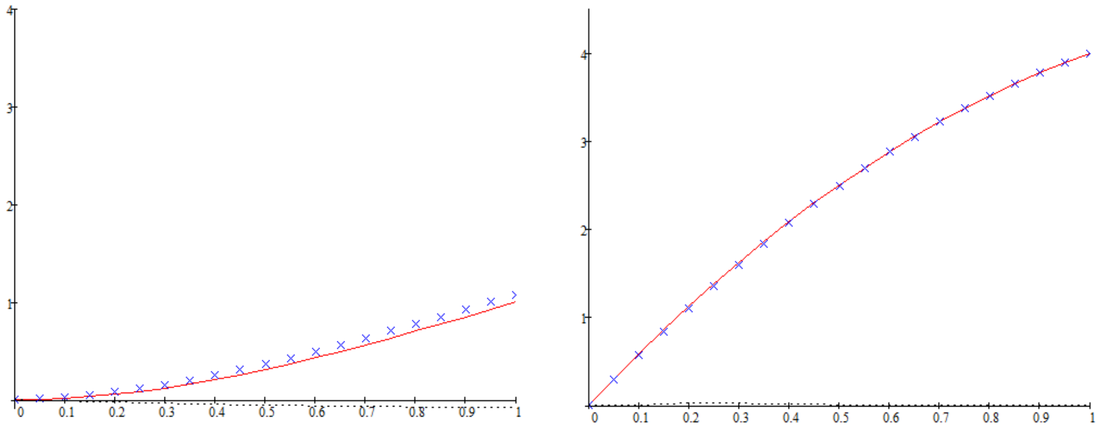


Fig. 1. Exact (solid line) and numerical (marked with $\times$) solutions and differences between exact and numerical solutions (marked with $\bullet$).

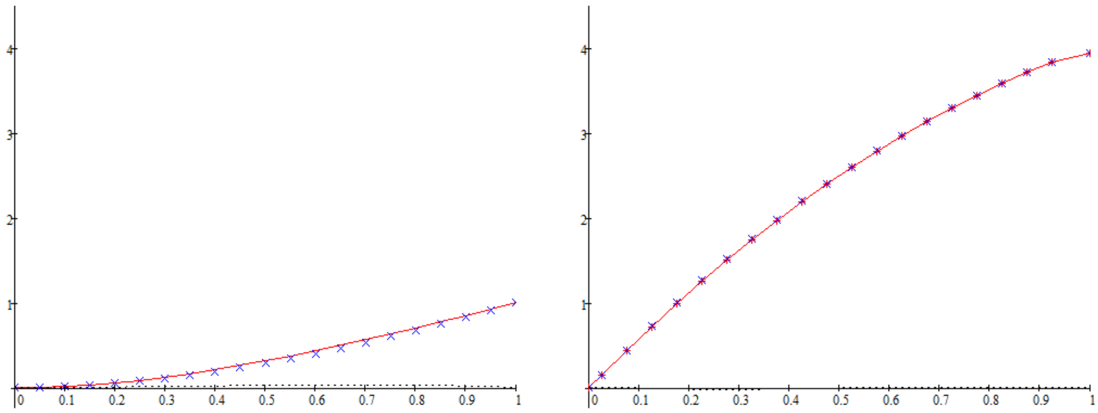


Fig. 2. Exact (solid line) and numerical (marked with $\times$) solutions and differences between exact and numerical solutions (marked with $\bullet$).

## R E F E R E N C E S

1. Mitchison G.J. A model for vein formation in higher plants. *Proc. R. Soc. Lond. B.*, **207**, 1166 (1980), 79-109.

2. Bell J., Cosner C., Bertiger W. Solutions for a flux-dependent diffusion model. *SIAM J. Math. Anal.*, **13**, 5 (1982), 758-769.

3. Dzhangveladze T.A. Averaged model of sum approximation for a system of nonlinear partial differential equations. (Russian) *Proc. I. Vekua Inst. Appl. Math.,* **19** (1987), 60-73.

4. Dzhangveladze T.A., Tagvarelia T.G. Convergence of a difference scheme for a system of nonlinear partial differential equations, that arise in biology. (Russian) *Proc. I. Vekua Inst. Appl. Math.,* **40** (1990), 77-83.

5. Jangveladze T.A. Investigation and numerical solution of some systems of nonlinear partial differential equations. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.,* **6**, 1 (1991), 25-28.

6. Jangveladze T., Kiguradze Z., Nikolishvili M. On investigation and numerical solution of one nonlinear biological model. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.,* **22** (2008), 46-50.

7. Jangveladze T., Kiguradze Z., Nikolishvili M. On aproximate solution of one nonlinear two-dimensional diffusion system. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.,* **23** (2009), 42-45.

8. Rheinboldt W.C. Methods for Solving Systems of Nonlinear Equations. SIAM, Philadelphia, 1970.

Received 13.05.2010; revised 15.09.2010; accepted 17.10.2010.

Authors' address:

M. Nikolishvili
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: maianikolishvili@yahoo.com