Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 24, 2010

# LIAPUNOV-TAUBER TYPE THEOREM IN THERMO-ELECTRO-MAGNETO-ELASTICITY THEORY

Mrevlishvili M.

**Abstract**. We prove a Liapunov-Tauber type theorem for the generalized double layer potential in the thermo-electro-magneto-elasticity theory.

**Keywords and phrases**: Potential method, layer potential, thermo-electro-magneto-elasticity.

**AMS subject classification**: 31B10, 74F05, 74F15.

Here we prove a Liapunov-Tauber type theorem for the generalized double layer potential in the thermo-electro-magneto-elasticity theory. This theorem plays a crucial role in the study of boundary value problems by the boundary integral equation method.

Similar theorem in the theory of harmonic functions states that the normal derivative of a harmonic double layer potential has no jump across the integration surface (see, e.g., [1], [2]). Analogous theorems for a double layer potential of the classical elasticity and thermoelasticity theory are also well known [3-5].

In this paper, we apply a quite different approach and present a very simplified proof of the Liapunov-Tauber type theorem for a sufficiently involved system of differential equations which model elastic solids with regard to electro-magnetic and thermal effects.

Throughout the paper $u = (u_1, u_2, u_3)^\top$ denotes the displacement vector, $\sigma_{ij}$ is the mechanical stress tensor, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ is the strain tensor, $E = (E_1, E_2, E_3)^\top$ and $H = (H_1, H_2, H_3)^\top$ are electric and magnetic fields respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, $\varphi$ and $\psi$ stand for the electric and magnetic potentials and $E = -\operatorname{grad} \varphi$, $H = -\operatorname{grad} \psi$; $\vartheta$ is the temperature increment, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and $\mathcal{S}$ is the entropy density.

First we present the field equations of the linear theory of thermo-electro-magneto-elasticity for anisotropic solids and introduce the corresponding matrix partial differential operators [6], [7]:

Constitutive relations:

$$\begin{aligned}
\sigma_{rj} &= \sigma_{jr} = c_{rjkl}\,\varepsilon_{kl} - e_{lrj}E_l - q_{lrj}H_l - \lambda_{rj}\vartheta, \qquad r, j = 1, 2, 3,\\
D_j &= e_{jkl}\,\varepsilon_{kl} + \varkappa_{jl}\,E_l + a_{jl}\,H_l + p_j\,\vartheta, \qquad j = 1, 2, 3,\\
B_j &= q_{jkl}\,\varepsilon_{kl} + a_{jl}\,E_l + \mu_{jl}H_l + m_j\,\vartheta, \qquad j = 1, 2, 3,\\
\mathcal{S} &= \lambda_{kl}\,\varepsilon_{kl} + p_k\,E_k + m_k\,H_k + \gamma\,\vartheta.
\end{aligned}$$

Fourier Law: $q_j = -\eta_{jl}\,\partial_l\vartheta, \quad j = 1, 2, 3.$
Equations of motion: $\partial_j\sigma_{rj} + X_r = \varrho\,\partial_t^2 u_r, \qquad r = 1, 2, 3.$

Quasi-static equations for electro-magnetic fields where the rate of magnetic field is small (electric field is curl free) and there is no electric current (magnetic field is curl free): $\partial_j D_j = \varrho_e \,,\ \partial_j B_j = 0\,.$
Linearized equation of the entropy balance: $T_0\, \partial_t \mathcal{S} - Q = -\partial_j q_j.$

Here and in what follows the summation over the repeated indices is meant from 1 to 3, unless stated otherwise; $\varrho$ is the mass density, $\varrho_e$ is the electric density, $c_{rjkl}$ are the elastic constants, $e_{jkl}$ are the piezoelectric constants, $q_{jkl}$ are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, $\mu_{jk}$ are the magnetic permeability constants, $a_{jk}$ are the coupling coefficients connecting electric and magnetic fields, $p_j$ and $m_j$ are constants characterizing the relation between thermodynamic processes and electromagnetic effects, $\lambda_{jk}$ are the thermal strain constants, $\eta_{jk}$ are the heat conductivity coefficients, $\gamma = \varrho\, c\, T_0^{-1}$ is the thermal constant, $T_0$ is the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, $c$ is the specific heat per unit mass, $X = (X_1, X_2, X_3)^\top$ is a mass force density, $Q$ is a heat source intensity. Further, let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with boundary $S \in C^{1,\kappa}$, $0 < \kappa \le 1$; $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$.

The corresponding homogeneous pseudo-oscillation equations of the thermo-electro-magneto-elasticity theory in matrix form read as

$$A(\partial, \tau)\, U(x) = 0\,, \tag{1}$$

where $U = (u_1, u_2, u_3, u_4, u_5, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top$, $\tau = \sigma + i\,\omega$ with $\sigma > 0$, and $A(\partial, \tau)$ is a nonselfadjoint strongly elliptic matrix differential operator,

$$A(\partial,\tau) := \begin{bmatrix} [c_{rjkl}\,\partial_j\,\partial_l - \varrho\,\tau^2\,\delta_{rk}]_{3\times 3} & [e_{lrj}\,\partial_j\partial_l]_{3\times 1} & [q_{lrj}\,\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\,\partial_j]_{3\times 1} \\ [-e_{jkl}\,\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\,\partial_j\,\partial_l & a_{jl}\,\partial_j\,\partial_l & -p_j\,\partial_j \\ [-q_{jkl}\,\partial_j\partial_l]_{1\times 3} & a_{jl}\,\partial_j\,\partial_l & \mu_{jl}\,\partial_j\,\partial_l & -m_j\,\partial_j \\ [-\tau\,T_0\,\lambda_{kl}\,\partial_l]_{1\times 3} & \tau\,T_0\,p_l\,\partial_l & \tau\,T_0\,m_l\,\partial_l & \eta_{jl}\,\partial_j\,\partial_l - \tau\,T_0\,\gamma \end{bmatrix}_{6\times 6}.$$

We introduce the generalized matrix stress operator

$$\mathcal{T}(\partial,n) := \begin{bmatrix} [c_{rjkl}\,n_j\,\partial_l]_{3\times 3} & [e_{lrj}\,n_j\,\partial_l]_{3\times 1} & [q_{lrj}\,n_j\,\partial_l]_{3\times 1} & [-\lambda_{rj}\,n_j]_{3\times 1} \\ [-e_{jkl}\,n_j\,\partial_l]_{1\times 3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l & -p_j\,n_j \\ [-q_{jkl}\,n_j\,\partial_l]_{1\times 3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l & -m_j\,n_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\,n_j\,\partial_l \end{bmatrix}_{6\times 6}.$$

For a six vector $U = (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial,n)\,U = (\,\sigma_{1j}\,n_j,\ \sigma_{2j}\,n_j,\ \sigma_{3j}\,n_j,\ -D_j\,n_j,\ -B_j\,n_j,\ -q_j\,n_j)^\top. \tag{2}$$

The components of the vector $\mathcal{T}\,U$ given by (2) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermo-electro-magneto-elasticity, the forth, fifth and sixth ones are respectively the normal components of the electric displacement vector, magnetic induction vector and heat flux vector with opposite sign.

We introduce also the boundary operator $\mathcal{P}(\partial, n, \tau)$ associated with the adjoint differential operator $A^*(\partial, \tau)$,

$$\mathcal{P}(\partial, n, \tau) = \begin{bmatrix} [c_{rjkl}\, n_j\, \partial_l]_{3\times 3} & [-e_{lrj}\, n_j\, \partial_l]_{3\times 1} & [-q_{lrj}\, n_j\, \partial_l]_{3\times 1} & [\overline{\tau}\, T_0\, \lambda_{rj}\, n_j]_{3\times 1} \\ [e_{jkl}\, n_j\, \partial_l]_{1\times 3} & \varkappa_{jl}\, n_j\, \partial_l & a_{jl}\, n_j\, \partial_l & -\overline{\tau}\, T_0\, p_j\, n_j \\ [q_{jkl}\, n_j\, \partial_l]_{1\times 3} & a_{jl}\, n_j\, \partial_l & \mu_{jl}\, n_j\, \partial_l & -\overline{\tau}\, T_0\, m_j\, n_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\, n_j\, \partial_l \end{bmatrix}_{6\times 6}.$$

Further we define the generalized single and double layer potentials

$$V(g)(x) = \int_S \Gamma(x-y, \tau)\, g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S,$$

$$W(g)(x) = \int_S [\mathcal{P}(\partial_y, n(y), \overline{\tau})\Gamma^\top(x-y, \tau)]^\top\, g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S,$$

where $g = (g_1, \cdots, g_6)^\top$ is a density vector-function defined on $S$ and $\Gamma(\cdot\,, \tau)$ is the fundamental matrix of the operator $A(\partial, \tau)$ constructed in [7].

With the help of the corresponding Green's identities, by standard arguments we can prove the following representation formula for a regular solution to equation (1)

$$W(\{U\}^+)(x) - V(\{\mathcal{T}U\}^+)(x) = \begin{cases} U(x) & \text{for} \quad x \in \Omega^+, \\ 0 & \text{for} \quad x \in \Omega^-. \end{cases}$$

Similar representation formula holds in the exterior domain $\Omega^-$ if a solution of the pseudo-oscillation equation satisfies the following decay conditions at infinity

$$u_k(x) = \mathcal{O}(|x|^{-2}),\ \varphi(x) = \mathcal{O}(|x|^{-1}),\ \psi(x) = \mathcal{O}(|x|^{-1}),\ \vartheta(x) = \mathcal{O}(|x|^{-2}),\ k = 1, 2, 3.$$

One can write then the following representation formula for a vector $U$ which is a solution of the homogeneous equation (1) in $\Omega^\pm$

$$U(x) = W([U]_S)(x) - V([\mathcal{T}U]_S)(x), \quad x \in \Omega^+ \cup \Omega^-, \tag{3}$$

where $[U]_S = \{U\}^+ - \{U\}^-$ and $[\mathcal{T}U]_S = \{\mathcal{T}U\}^+ - \{\mathcal{T}U\}^-$ on $S$.

Now we formulate the Lyapunov-Tauber type theorem in the thermo-electro-magneto-elasticity theory.

**Theorem.** *Let $S \in C^{2,\kappa}$, $0 < \kappa < 1$, and $h \in [C^{1,\kappa'}(S)]^6$ with $0 < \kappa' < \kappa$. For all $x \in S$ there holds the following equality*

$$[\,\mathcal{T}(\partial_x, n(x))\, W(h)(x)\,]^+ = [\,\mathcal{T}(\partial_x, n(x))\, W(h)(x)\,]^- =: \mathcal{L}\, h(x),$$

*where the symbols $[\,\cdot\,]^\pm$ denote the one-sided limits on $S$ from $\Omega^\pm$ respectively.*

**Proof.** First of all let us note that for any $g \in [C^{0,\kappa'}(S)]^6$ and for all $x \in S$ we have the following jump relations for the single and double layer potentials (for details see [7])

$$[\,V(g)(x)\,]^\pm = V(g)(x) = \mathcal{H}\, g(x), \tag{4}$$

$$[\,\mathcal{T}(\partial_x, n(x))\, V(g)(x)\,]^\pm = [\,\mp 2^{-1} I_6 + \mathcal{K}\,]\, g(x), \tag{5}$$

$$[\,W(g)(x)\,]^\pm = [\,\pm 2^{-1} I_6 + \mathcal{N}\,]\, g(x), \tag{6}$$

where

$$\mathcal{H}\,g(x) := \int_S \Gamma(x-y,\tau)\,g(y)\,dS_y\,,$$
$$\mathcal{K}\,g(x) := \int_S [\,\mathcal{T}(\partial_x,n(x))\,\Gamma(x-y,\tau)\,]\;g(y)\,dS_y\,,$$
$$\mathcal{N}\,g(x) := \int_S [\,\mathcal{P}(\partial_y,n(y),\overline{\tau})\,\Gamma^\top(x-y,\tau)\,]^\top\,g(y)\,dS_y\,.$$

The operator $\mathcal{H}$ is weakly singular, while $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators. Moreover, the operators

$$\begin{array}{rlrl} V &: [C^{1,\kappa'}(S)]^6 \to [C^{2,\kappa'}(\overline{\Omega^\pm})]^6, & W &: [C^{1,\kappa'}(S)]^6 \to [C^{1,\kappa'}(\overline{\Omega^\pm})]^6, \\ \mathcal{H} &: [C^{1,\kappa'}(S)]^6 \to [C^{2,\kappa'}(S)]^6, & \mathcal{K},\mathcal{N} &: [C^{1,\kappa'}(S)]^6 \to [C^{1,\kappa'}(S)]^6, \end{array} \tag{7}$$

are continuous.

Now let us consider the double layer potential $U(x) := W(h)(x)$ with $h \in [C^{1,\kappa'}(S)]^6$. Clearly $U = W(h) \in [C^{1,\kappa'}(\overline{\Omega^\pm})]^6$ due to the mapping properties (7) and since $S \in C^{2,\kappa}$. Consequently, the limits $[\,\mathcal{T}(\partial_x,n(x))\,U(x)\,]^\pm \equiv [\,\mathcal{T}(\partial_x,n(x))\,W(h)(x)\,]^\pm$ exist. In accordance with the above jump relations (4)-(6) and the representation forma (3) we then derive $U(x) = W([U]_S)(x) - V([\mathcal{T}U]_S)(x), \quad x \in \Omega^\pm$, i.e.,

$$W(h)(x) = W(h)(x) - V([\mathcal{T}W(h)]_S)(x), \quad x \in \Omega^\pm,$$

since $[U]_S = \{W(h)\}^+ - \{W(h)\}^- = h$ on $S$ due to (6). Therefore $V([\mathcal{T}W(h)]_S) = 0$ in $\Omega^\pm$ and in view of (5) we conclude

$$\{\mathcal{T}V([\mathcal{T}W(h)]_S)\}^- - \{\mathcal{T}V([\mathcal{T}W(h)]_S)\}^+ = [\mathcal{T}W(h)]_S = \{\mathcal{T}W(h)\}^+ - \{\mathcal{T}W(h)\}^- = 0$$

on $S$, which completes the proof. □

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Received 8.06.2010; revised 19.09.2010; accepted 21.10.2010.

Author's address:

M. Mrevlishvili
Department of Mathematics
Georgian Technical University
77, M. Kostava St., Tbilisi 0175
Georgia
E-mail: mmrevlishvili@dls.ge