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ON ONE APPLICATION I. VEKUA'S NORMED MOMENTS METHOD IN THEORY OF PLATES

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Abstract. In this paper by I. Vekua's approximate of order $N = 3$ the problem of stress concentration (Kirsch) are solved.

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1. Making use of vector and tensor notations the equilibrium equation of the 3D elastic bodies and stress-strain relations with respect to the Cartesian coordinates can be written as follows:

$$\partial_i \boldsymbol{\sigma}^i + \boldsymbol{\Phi} = 0, \quad \boldsymbol{\sigma}^i = \sigma^{ij}\mathbf{e_j}, \quad \sigma^{ij} = E^{ijpq}e_{pq}, \quad (\mathbf{e_j} = \partial_\mathbf{j}\mathbf{R}),$$

$$E^{ijpq} = \lambda\delta^{ij}\delta^{pq} + \mu(\delta^{ip}\delta^{jq} + \delta^{iq}\delta^{jp}),$$

$$2e_{pq} = \mathbf{e_q}\partial_p\mathbf{u} + \mathbf{e_p}\partial_q\mathbf{u} \quad (i,j,p,q = 1,2,3),$$

where $\boldsymbol{\sigma}^i = \boldsymbol{\sigma}_i$ are constituents of the stress tensor, $\boldsymbol{\Phi}$ is vector of volume force, σ^{ij} and e^{pq} are components of the stress and strain tensors, $\mathbf{u}$ is the displacement vector, $\mathbf{R}$ is radius-vector of the point $M(x_1, x_2, x_3)$, δ^{ij} are the Kronecker symbol. Under a repeated indices we mean summation, note that the Greek indices range over $1, 2$, while Latin indices range over $1, 2, 3$.

By means of Vekua normed moments method the 2-D equilibrium equation and stress-strain relations takes the form [1]:

$$\partial_\alpha \overset{(m)}{\boldsymbol{\sigma}}{}^\alpha - \frac{1}{h}\overset{(m)}{\underline{\boldsymbol{\sigma}}}{}^3 + \overset{(m)}{\mathbf{F}} = 0, \quad \overset{(m)}{\boldsymbol{\sigma}}{}^i = \underset{\sim}{\overset{(m)}{\boldsymbol{\sigma}}}{}^i + \overset{(m)}{\boldsymbol{X}}{}^i \quad (m = 0, 1, ..., N), \tag{1}$$

where

$$\underset{\sim}{\overset{(m)}{\boldsymbol{\sigma}}}{}^i = \lambda\left(D_j \overset{(m)}{u}{}^j\right)\mathbf{e^i} + \mu\left(D_i \overset{(m)}{u}_j + D_j \overset{(m)}{u}_i\right)\mathbf{e^j} - \varepsilon_{N,m}\sum_{s=0}^{N}(1 + (-1)^{s+m})\times$$

$$\left[\left(\frac{\lambda}{\lambda+2\mu}D_\alpha \overset{(s)}{u}{}^\alpha + D_3\overset{(s)}{u}{}^3\right)\left(\lambda\mathbf{e^i} + \mu\delta^{i3}\mathbf{e_3}\right) + \mu\left(D_3\overset{(s)}{u}{}^\alpha + D_\alpha\overset{(s)}{u}_3\right)\left(\delta^{i\alpha}\mathbf{e_3} + \delta^{i3}\mathbf{e}^\alpha\right)\right], \tag{2}$$

$$\overset{(m)}{\underline{\boldsymbol{\sigma}}}{}^3 = (2m+1)\left(\overset{(m-1)}{\boldsymbol{\sigma}}{}^3 + \overset{(m-3)}{\boldsymbol{\sigma}}{}^3 + \cdots\right), \quad \overset{(m)}{\mathbf{F}} = \overset{(m)}{\boldsymbol{\Phi}} + \frac{2m+1}{2h}\left(\overset{(+)}{\boldsymbol{\sigma}}{}^3 - (-1)^m\overset{(-)}{\boldsymbol{\sigma}}{}^3\right),$$

$$\overset{(\pm)}{\boldsymbol{\sigma}}{}^3 = \boldsymbol{\sigma}^3(x_1, x_2, \pm h) = \mp\overset{(\pm)}{\mathbf{P}},$$

$$
\begin{aligned}
&\overset{(m)}{\mathbf{X}^{\mathbf{i}}}=-\varepsilon_{_{N,m}}\Bigg[\Big(\overset{(+)}{P^{\alpha}}-(-1)^m\overset{(-)}{P^{\alpha}}\Big)\Big(\delta^i_\alpha\mathbf{e_3}+\delta^{i3}\mathbf{e}_\alpha\Big)\\
&+\frac{1}{\lambda+2\mu}\Big(\overset{(+)}{P_3}-(-1)^m\overset{(-)}{P_3}\Big)\Big(\lambda\mathbf{e^i}+2\mu\delta^{i3}\mathbf{e_3}\Big)\Bigg],\\
&\Big(\overset{(m)}{\boldsymbol{\sigma}}{}^i,\overset{(m)}{\mathbf{u}},\overset{(m)}{\mathbf{\Phi}}\Big)=\frac{2m+1}{2h}\int\limits_{-h}^{h}\Big(\boldsymbol{\sigma}^i,\mathbf{u},\mathbf{\Phi}\Big)P_m\Big(\frac{x_3}{h}\Big)dx_3,\\
&\varepsilon_{_{N,m}}=\frac{2m+1}{N(N+2)}\Big(1-\frac{(-1)^{m+N}}{N+1}\Big),\\
&D_\alpha\overset{(m)}{\mathbf{u}}=\partial_\alpha\overset{(m)}{\mathbf{u}},\quad D_3\overset{(m)}{\mathbf{u}}=\frac{2m+1}{h}\sum_{s=m}^{N}\frac{1-(-1)^{s+m}}{2}\overset{(s)}{\mathbf{u}}.
\end{aligned}
$$

Note that the scheme of construction of the approximate solution

$$
\begin{aligned}
&\mathbf{u}(x_1,x_2,x_3)=\sum_{m=0}^{N}\overset{(m)}{\mathbf{u}}(x_1,x_2)P_m\Big(\tfrac{x_3}{h}\Big),\quad(-h\le x_3\le h),\\
&\boldsymbol{\sigma}^i(x_1,x_2,x_3)=\sum_{m=0}^{N}\overset{(m)}{\boldsymbol{\sigma}}{}^i(x_1,x_2)P_m\Big(\tfrac{x_3}{h}\Big),
\end{aligned}
$$

of equilibrium problems are compatible with boundary data of face surface $x_3=\pm h$, where $P_m\Big(\frac{x_3}{h}\Big)$ is Legendre polynomial of order m, i.e.

$$
\overset{(\pm)}{\boldsymbol{\sigma}}{}^3=\sum_{m=0}^{N}(\pm1)^m\overset{(m)}{\boldsymbol{\sigma}}_3=\mp\overset{(\pm)}{\mathbf{P}}.
$$

The equations (1), (2) constitute a normal system of $(6N+6)$ order if the conditions

$$
1-2\varepsilon_{_{N,m}}\neq0\quad(m=0,1,...,N)
$$

are satisfied. As regards the cases $N=0,1,2$ these systems are degenerated and need special examination. I. Vekua explicitly considered these systems and pointed out the class of problems which are solvable by means of these systems of equations.

2. The equilibrium equation and stress-strain relations for the approximation $N=3$ has the complex form:

a) Equilibrium equations

$$
\begin{aligned}
&\partial_z\Big(\overset{(0)}{\underset{\sim}{\sigma}}_{11}-\overset{(0)}{\underset{\sim}{\sigma}}_{22}+2i\overset{(0)}{\underset{\sim}{\sigma}}_{12}\Big)+\partial_{\bar z}\Big(\overset{(0)}{\underset{\sim}{\sigma}}_{11}+\overset{(0)}{\underset{\sim}{\sigma}}_{22}\Big)+\overset{(0)}{Y}_+=0,\\
&\partial_z\Big(\overset{(2)}{\underset{\sim}{\sigma}}_{11}-\overset{(2)}{\underset{\sim}{\sigma}}_{22}+2i\overset{(2)}{\underset{\sim}{\sigma}}_{12}\Big)+\partial_{\bar z}\Big(\overset{(2)}{\underset{\sim}{\sigma}}_{11}+\overset{(2)}{\underset{\sim}{\sigma}}_{22}\Big)-\tfrac{5}{h}\overset{(1)}{\underset{\sim}{\sigma}}_{+}+\overset{(2)}{Y}_+=0,\\
&\partial_z\overset{(1)}{\underset{\sim}{\sigma}}_{+}+\partial_{\bar z}\overset{(1)}{\underset{\sim}{\bar\sigma}}_{+}-\tfrac{3}{h}\overset{(0)}{\underset{\sim}{\sigma}}_{33}+\overset{(1)}{Y}_3=0,\\
&\partial_z\overset{(3)}{\underset{\sim}{\sigma}}_{+}+\partial_{\bar z}\overset{(3)}{\underset{\sim}{\bar\sigma}}_{+}-\tfrac{7}{h}\Big(\overset{(0)}{\underset{\sim}{\sigma}}_{33}+\overset{(2)}{\underset{\sim}{\sigma}}_{33}\Big)+\overset{(3)}{Y}_3=0,\quad(2\partial_z=\partial_1-i\partial_2),
\end{aligned}
$$

where

$$
\begin{aligned}
&\overset{(m)}{Y}_{+}=\overset{(m)}{Y}_{1}+i\overset{(m)}{Y}_{2},\quad (m=0,2),\\
&\overset{(0)}{Y}_{\alpha}=-\frac{1}{12}\frac{\lambda}{\lambda+2\mu}\partial_\alpha\left(\overset{(+)}{P}_{3}-\overset{(-)}{P}_{3}\right)-\frac{1}{2h}\left(\overset{(+)}{P}_{\alpha}+\overset{(-)}{P}_{\alpha}\right)+\overset{(0)}{\Phi}_{\alpha},\\
&\overset{(2)}{Y}_{\alpha}=-\frac{5}{12}\frac{\lambda}{\lambda+2\mu}\partial_\alpha\left(\overset{(+)}{P}_{3}-\overset{(-)}{P}_{3}\right)-\frac{7}{4h}\left(\overset{(+)}{P}_{\alpha}+\overset{(-)}{P}_{\alpha}\right)+\overset{(2)}{\Phi}_{\alpha},\\
&\overset{(1)}{Y}_{3}=-\frac{3}{20}\frac{\lambda}{\lambda+2\mu}\partial_\alpha\left(\overset{(+)}{P}_{\alpha}+\overset{(-)}{P}_{\alpha}\right)-\frac{5}{4h}\left(\overset{(+)}{P}_{3}-\overset{(-)}{P}_{3}\right)+\overset{(1)}{\Phi}_{\alpha},\\
&\overset{(3)}{Y}_{3}=-\frac{7}{20}\partial_\alpha(\overset{(+)}{P}_{\alpha}+\overset{(-)}{P}_{\alpha}\Big)+\overset{(3)}{\Phi}_{3}.
\end{aligned}
$$

b) Hooke's law

$$
\begin{aligned}
&\underset{\sim}{\overset{(0)}{\sigma}}_{11}-\underset{\sim}{\overset{(0)}{\sigma}}_{22}+2i\underset{\sim}{\overset{(0)}{\sigma}}_{12}=4\mu\partial_{\bar z}\overset{(0)}{u}_{+}\quad\left(\overset{(k)}{u}_{+}=\overset{(k)}{u}_{1}+i\overset{(k)}{u}_{2},k=0,2\right),\\
&\underset{\sim}{\overset{(0)}{\sigma}}_{11}+\underset{\sim}{\overset{(0)}{\sigma}}_{22}=\left(2\mu+\frac{\lambda}{3}\frac{5\lambda+12\mu}{\lambda+2\mu}\right)\overset{(0)}{\theta}-\frac{\lambda^2}{3(\lambda+2\mu)}\overset{(2)}{\theta}+\frac{5\lambda}{3h}\overset{(1)}{u}_{3},\\
&\underset{\sim}{\overset{(2)}{\sigma}}_{11}-\underset{\sim}{\overset{(2)}{\sigma}}_{22}+2i\underset{\sim}{\overset{(2)}{\sigma}}_{12}=4\mu\partial_{\bar z}\overset{(2)}{u}_{+}\quad\left(\overset{(k)}{\theta}=\partial_\alpha\overset{(k)}{u}^{\alpha},k=0,2\right),\\
&\underset{\sim}{\overset{(2)}{\sigma}}_{11}+\underset{\sim}{\overset{(2)}{\sigma}}_{22}=\left(2\mu+\frac{\lambda}{3}\frac{5\lambda+12\mu}{\lambda+2\mu}\right)\overset{(2)}{\theta}-\frac{5\lambda^2}{3(\lambda+2\mu)}\overset{(0)}{\theta}-\frac{5\lambda}{3h}\overset{(1)}{u}_{3},\\
&\underset{\sim}{\overset{(1)}{\sigma}}_{+}=\underset{\sim}{\overset{(1)}{\sigma}}_{13}+i\underset{\sim}{\overset{(1)}{\sigma}}_{23}=\frac{\mu}{10}\left[2\partial_{\bar z}\left(7\overset{(1)}{u}_{3}-3\overset{(3)}{u}_{3}\right)+\frac{21}{h}\overset{(2)}{u}_{+}\right]=-\underset{\sim}{\overset{(3)}{\sigma}}_{+},\\
&\underset{\sim}{\overset{(0)}{\sigma}}_{33}=\frac{1}{6}\left[\lambda\left(5\overset{(0)}{\theta}-\overset{(2)}{\theta}\right)+\frac{5(\lambda+2\mu)}{h}\overset{(1)}{u}_{3}\right]=-\underset{\sim}{\overset{(2)}{\sigma}}_{33}.
\end{aligned}
$$

c) Equilibrium equations with respect to components of the displacement vector

$$
\begin{aligned}
&\mu\Delta\overset{(0)}{u}_{+}+\partial_{\bar z}\left[\left(2\mu+\frac{\lambda}{3}\frac{5\lambda+12\mu}{\lambda+2\mu}\right)\overset{(0)}{\theta}-\frac{\lambda^2}{3(\lambda+2\mu)}\overset{(2)}{\theta}+\frac{5\lambda}{3h}\overset{(1)}{u}_{3}\right]+\overset{(0)}{Y}_{+}=0,\\
&\mu\Delta\overset{(2)}{u}_{+}+\partial_{\bar z}\left[\left(2\mu+\frac{\lambda}{3}\frac{5\lambda+12\mu}{\lambda+2\mu}\right)\overset{(2)}{\theta}-\frac{5\lambda^2}{3(\lambda+2\mu)}\overset{(0)}{\theta}-\frac{5\lambda}{3h}\overset{(1)}{u}_{3}\right]\\
&-\frac{\mu}{h}\left[\partial_{\bar z}\left(7\overset{(1)}{u}_{3}-3\overset{(3)}{u}_{3}\right)+\frac{21}{h}\overset{(2)}{u}_{+}\right]+\overset{(2)}{Y}_{+}=0,\\
&\frac{\mu}{10}\left[\Delta\left(7\overset{(1)}{u}_{3}-3\overset{(3)}{u}_{3}\right)+\frac{21}{h}\overset{(2)}{u}_{+}\right]-\frac{\lambda}{2h}\left(5\overset{(0)}{\theta}-\overset{(2)}{\theta}\right)-\frac{5(\lambda+2\mu)}{2h^2}\overset{(1)}{u}_{3}+\overset{(1)}{Y}_{+}=0,\\
&-\frac{\mu}{10}\left[\Delta\left(7\overset{(1)}{u}_{3}-3\overset{(3)}{u}_{3}\right)+\frac{21}{h}\overset{(2)}{u}_{+}\right]+\overset{(3)}{Y}_{+}=0.
\end{aligned}
\tag{3}
$$

The complex representation of the general solution has the form

$$
\begin{aligned}
&\overset{(0)}{u}_+ = \varkappa^*\varphi_1(z) - z\overline{\varphi_1(z)} - \overline{\psi_1(z)}, \qquad \left(\varkappa^* = \frac{5\lambda+6\mu}{3\lambda+2\mu}\right),\\
&\overset{(2)}{u}_+ = \varphi_2(z) + z\overline{\varphi_2'(z)} + \overline{\psi_2'(z)} + \frac{4\lambda h^2}{3(3\lambda+2\mu)}\overline{\varphi_1''(z)} + \frac{4h^2(33\lambda+40\mu)}{105(\lambda+2\mu)}\overline{\varphi_2''(z)} + \frac{ih}{21}\partial_{\bar z}\omega,\\
&\overset{(1)}{u}_3 = \frac{2\lambda h}{5(\lambda+2\mu)}\left[\varphi_2'(z) + \overline{\varphi_2'(z)} - \frac{5(\lambda+2\mu)}{3\lambda+2\mu}\Big(\varphi_1'(z) + \overline{\varphi_1'(z)}\Big)\right],\\
&\overset{(3)}{u}_3 = \frac{7}{2h}\left[\bar z\varphi_2(z) + z\overline{\varphi_2(z)} + \psi_2(z) + \overline{\psi_2(z)}\right],\\
&\Delta\omega - \frac{21}{2h^2}\omega = 0,\\
&\omega = Re\left[f(z) - \int_{z_0}^{z} f(t)\frac{\partial}{\partial t}I_0\left(\sqrt{\frac{21}{2}}\frac{1}{h}\sqrt{(z-t)(\bar z - \bar z_0)}\right)dt\right],
\end{aligned}
$$

where $f(z), \varphi_1(z), \varphi_2(z), \psi_1(z), \psi_2(z)$ are arbitrary analytic functions of $z = x_1 + ix_2$, I_0 is Besels's function of the first kind of zero order with on imaginary argument.

3. Now consider the stress concentrations problem of the infinity plate weakened circular hole (problem of E. Kirsch [2]).

For the case the conditions of infinite may be written as:

$$
\left(\overset{(0)}{\sigma}_{11}\right)^{\infty} = P = const, \qquad \left(\overset{(0)}{\sigma}_{12} = \overset{(0)}{\sigma}_{22} = \cdots = \overset{(3)}{\sigma}_{13}\right)^{\infty} = 0. \tag{4}
$$

The boundary conditions on the contour $|z| = R$ have the form:

$$
\overset{(0)}{\sigma}_{rr} + i\overset{(0)}{\sigma}_{r\vartheta} = 0, \quad \overset{(2)}{\sigma}_{rr} + i\overset{(2)}{\sigma}_{r\vartheta} = 0, \quad \overset{(1)}{\sigma}_{r3} = \overset{(3)}{\sigma}_{r3} = 0. \tag{5}
$$

The solution of the boundary value problem (3), (4), (5) may be represented by series of the form:

$$
\begin{aligned}
&f(z) = \sum_{n=0}^{\infty}\frac{a_n}{z^n}, \quad \varphi_1'(z) = \sum_{n=0}^{\infty}\frac{\overset{(1)}{a}_n}{z^n}, \quad \psi_1'(z) = \sum_{n=1}^{\infty}\frac{\overset{(1)}{b}_n}{z^n},\\
&\varphi_2'(z) = \sum_{n=1}^{\infty}\frac{\overset{(2)}{a}_n}{z^n}, \quad \psi_2''(z) = \sum_{n=1}^{\infty}\frac{\overset{(2)}{b}_n}{z^n}, \quad \omega = \sum_{n=-\infty}^{\infty} c_n K_n(\gamma r)e^{in\vartheta},\\
&\left(\gamma^2 = \frac{21}{2h^2}, \ K_n(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left(1 + O\left(\frac{1}{x}\right)\right)\right).
\end{aligned}
$$

Making use of the conditions (4) and (5), we get

$$
\begin{aligned}
&\varphi_1'(z) = \frac{P}{8\mu}\left(1 - \frac{2R^2}{z^2}\right), \quad \psi_1'(z) = -\frac{P}{4\mu}\left(1 - \frac{R^2}{z^2} + \frac{3R^4}{z^4}\right),\\
&\varphi_2'(z) = 0, \quad \psi_2''(z) = \frac{2P\lambda R^2h^2}{\mu(\lambda+2\mu)}\frac{1}{z^4}, \quad \omega = 0.
\end{aligned}
$$

For the concentration coefficient of stresses we have [2, p.194]

$$K = \max_{|z|=R} \frac{\sigma_{\vartheta\vartheta}}{P} = 3,$$

which coincide with the classical coefficient of Kirsch.

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