Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 24, 2010

ON SOME GENERALIZATION OF DIRICHLET BOUNDARY CONDITION

Lobjanidze G.

Abstract. The family of boundary problems that consist the homogeneous Dirichlet and Neumann classical boundary conditions and one kind of nonlocal integral condition is considered. The variational formulation of suitable boundary problem on rectangle domains for Poisson's equation is studied.

Keywords and phrases: Dirichlet condition; nonlocal integral condition; variational formulation.

AMS subject classification: 35J25; 35J20; 35B99.

Let us introduce some notations and definitions. By ∂G we denote boundary of the rectangle $G = \{(x,y) |\ -a < x < 0,\ \ 0 < y < b\}$, where a and b are the given positive constants. Γ_t is the intersection of the line $x = t$ with the $\overline{G} = G \cup \partial G$.

We consider following problem: Find the function $u \in C^{(2)}(G) \cap C(\overline{G})$ satisfying Poisson's equation with classical boundary and one kind of nonlocal integral conditions:

$$-\Delta u(x,y) = f(x,y), \quad (x,y) \in G, \tag{1}$$

$$u(-a,y) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,b) = 0, \tag{2}$$

$$u(0,y) = \int\limits_0^b \int\limits_{-a}^0 \rho(x) u(x,y)\,dxdy, \quad x \in \,]-a,0[\,, \quad y \in \,]0,b[\,. \tag{3}$$

Here $f \in C(\overline{G})$ and $\rho \in C[-a,0]$.

Many scientists have been investigating nonlocal boundary value problems for ordinary differential equations and partial differential elliptic equations (see, for example, [1-17] and references therein).

From (3) we have that unknown function u is constant on Γ_0. Let us introduce the space

$$V = \{v |\quad v \in W_2^1(G), \quad v(-a,y) = 0, \quad v(0,y) = const\}.$$

Here $v(-a,y)$ and $v(0,y)$ are traces of the function $v \in W_2^1(G)$ on Γ_{-a} and Γ_0 respectively. The scalar product in V is induced from $W_2^1(G)$ space. Here V is the Hilbert space and if ℓ is the linear continuous functional defined on V and the Ritz representation takes place: it exists unique $\varphi_\ell \in V$ such that for $\forall v \in V$ we have

$$\ell v = (\varphi_\ell, v) = \int\limits_0^b \int\limits_{-a}^0 \left(\varphi_\ell(x,y) v(x,y) + \frac{\partial \varphi_\ell(x,y)}{\partial x} \frac{\partial v(x,y)}{\partial x} + \frac{\partial \varphi_\ell(x,y)}{\partial y} \frac{\partial v(x,y)}{\partial y} \right) dxdy.$$

The functions of the space V that for each linear continuous functional satisfy boundary condition $\ell(v) = v(0,y)$ or

$$\int_0^b \int_{-a}^0 \left(\varphi_\ell(x,y)v(x,y) + \frac{\partial \varphi_\ell(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x} + \frac{\partial \varphi_\ell(x,y)}{\partial y}\frac{\partial v(x,y)}{\partial y} \right) dxdy = v(0,y) \quad (4)$$

define subspace of the V. We denote this subspace by

$$V_\ell = \{v| \ v \in V, \quad \ell(v) = v(0,y)\}.$$

For example, if ℓ is the functional that corresponds to each function of V its constant trace value on Γ_0 (this functional is continuous after theorem of trace) then condition (4) is fulfilled for all function of the space V and V_ℓ coincides with V. Now if we take in a role of ℓ the considered functional multiplied on constant not equal to one then V_ℓ coincides with the subspace the traces of all elements of which is zero on Γ_0.

Let us take function $\varphi_\ell(x,y) = \varphi(x)$, which satisfies condition (4) and which is the solution of the following boundary value problem:

$$\begin{gathered} -\varphi''(x) + \varphi(x) = \rho(x), \quad x \in]-a,0[\,, \\ \varphi(-a) = 0, \quad \varphi'(0) = p, \quad p \in R. \end{gathered} \quad (5)$$

The condition (4) becomes as

$$\int_0^b \int_{-a}^0 \rho(x)v(x,y)dxdy + pbv(0,y) = v(0,y),$$

which gives nonlocal condition (3) if $p = 0$. Let us note that abovementioned examples will be receive if we take in (5) $\rho(x) \equiv 0$, $p = 1/b$ and $\rho(x) \equiv 0$, $p \neq 1/b$.

Let us denote by $D(\overline{G})$ the lineal of all the real functions v satisfying the following conditions:

1. v is defined almost everywhere on $\overline{G}$, and the boundary value $v(0,y)$ (the value on the boundary Γ_0) is equivalent to constant function defined on Γ_0, that is where is a $v_0 \in R$ such that $v(0,y) = v_0$ almost everywhere on $y \in [0,b]$;

2. $v \in L_2(G)$, $v_0 = \displaystyle\int_0^b \int_{-a}^0 \rho(x)v(x,y)dxdy.$

Two functions v and w are assumed as the same element of $D(\overline{G})$ if $v(x,y) = w(x,y)$ almost everywhere on $\overline{G}\backslash\Gamma_0$ and $v_0 = w_0$.

Let us $\overline{Q}$ is the closer of the rectangle $Q = \{(x,y)| \ 0 < x < a, \ 0 < y < b\}$ and define on $D(\overline{G})$ the operator of symmetrical extension τ as follows:

$$(\tau v)(x,y) = \begin{cases} v(x,y), & (x,y) \in \overline{G}, \\ -v(x,y) + 2v(0,y), & (x,y) \in \overline{Q}. \end{cases}$$

For two arbitrary functions v_1 and v_2 from the lineal $D(\overline{G})$ we define the scalar product

$$[v_1, \ v_2] = \int_0^b \int_{-a}^a \int_{-a}^x \widetilde{v}_1(s,y)\widetilde{v}_2(s,y)dsdxdy,$$

where $\widetilde{v}_i(s,y) = (\tau v_i)(s,y), \quad i = 1,2.$

After the introduction of the scalar product the lineal $D(\overline{G})$ becomes the pre-Hilbert space, which we denote by $H(\overline{G})$. The norm originated from this scalar product we denote by $\|\cdot\|_H$:

$$\|v\|_H^2 = \int_0^b \int_{-a}^a \int_{-a}^x \widetilde{v}^2(s,y)dsdxdy.$$

Theorem 1. *The norm defined on the $H(\overline{G})$ by the formula*

$$\|v\|^2 = \|v\|_{L_2(G)}^2 + v_0^2$$

is equivalent to the norm $\|\cdot\|_H$.

Consequence. *$H(\overline{G})$ is the Hilbert space.*

Let, the area of definition of the operator $A = -\Delta$ is the lineal $D_A(\overline{G})$ of the functions v defined on $\overline{G}$ for which the following conditions are fulfilled:

1. $v \in C^{(\infty)}(\overline{G})$, $\quad \dfrac{\partial^k v}{\partial x^k}(0,y) = 0, \quad \forall y \in [0,b]$,
$\dfrac{\partial^k v}{\partial y^k}(x,0) = \dfrac{\partial^k v}{\partial y^k}(x,b) = 0, \;\; \forall x \in [-a,0]\,, \;\; k = 1,\; 2,\; ...;$

2. $v(-a,y) = 0, \quad v(0,y) = v_0 = \displaystyle\int_0^b \int_{-a}^0 \rho(x)v(x,y)dxdy, \;\; \forall y \in [0,b]\,.$

Theorem 2. *The lineal $D_A(\overline{G})$ is dense in the space $H(\overline{G})$.*

Theorem 3. *There exists constant $\gamma > 0$ such that if*

$$\int_{-a}^0 \rho^2(x)dx < \gamma, \tag{6}$$

when A is positively defined on the lineal $D_A(\overline{G})$.

So, if the condition (6) is fulfilled then A is positive definite operator defined on the lineal $D_A(\overline{G})$ which is dense in the space $H(\overline{G})$ and for problem (1)-(3) we can use the standard way of the variational formulation [18]. Let us introduce the new scalar product on $D_A(\overline{G})$:

$$\begin{gathered}[v_1,v_2]_A = [Av_1,v_2] = \int_0^b \int_{-a}^a \int_{-a}^x \left(\frac{\partial \widetilde{v}_1(s,y)}{\partial s} \frac{\partial \widetilde{v}_2(s,y)}{\partial s} \right. \\ \left. + \frac{\partial \widetilde{v}_1(s,y)}{\partial y} \frac{\partial \widetilde{v}_2(s,y)}{\partial y} \right) dsdxdy - 2bv_1(0,y)v_2(0,y). \end{gathered} \tag{7}$$

For corresponding norm we use the notation $\|\cdot\|_A$.

After introducing the scalar product (7) the lineal $D_A(\overline{G})$ becomes the pre-Hilbert space which we denote by $S_A(\overline{G})$. By $H_A(\overline{G})$ we denote the Hilbert space obtained after completion of $S_A(\overline{G})$ by the norm $\|\cdot\|_A$.

Theorem 4. *The $\|\cdot\|_A$ and $\|\cdot\|_{W_2^1(G)}$ defined in the space $S_A(\overline{G})$ are equivalent norms.*

So,

$$H_A(\overline{G}) = \left\{ v| \;\; v \in W_2^1(G), \;\; v(-a,y) = 0, \;\; v(0,y) = \int_0^b \int_{-a}^0 \rho(x)v(x,y)dxdy \right\}.$$

For every function $f \in H(\overline{G})$ the quadratic functional

$$Fv = [v, v]_A - 2\,[f, v] \tag{8}$$

has the unique function $u \in H_A(\overline{G})$, which minimizes the functional (8) and satisfies the identity

$$[v, v]_A = [f, v]$$

for every $v \in H_A(\overline{G})$.

The functional (8) in the extended form can be written as

$$\begin{gathered} Fv = 2a \int\limits_0^b \int\limits_{-a}^0 \left[\left(\frac{\partial v(x,y)}{\partial x} \right)^2 + \left(\frac{\partial v(x,y)}{\partial y} \right)^2 - 2f(x,y)v(x,y) \right] dxdy- \\ -2bv_0^2 + 4 \int\limits_0^b \int\limits_{-a}^0 \int\limits_x^0 (v_0 f(s,y) + f_0 v(s,y)) dsdy - 4a^2 b f_0 v_0. \end{gathered} \tag{9}$$

Theorem 5. *The condition*

$$f_0 = \int\limits_0^b \int\limits_{-a}^0 \rho(x) f(x,y) dxdy = 0 \tag{10}$$

is necessary and sufficient in order to the solution of problem (1)-(3) $u \in H_A(\overline{G})$ minimizes the functional (9).

Let us note that if $f(x, y)$ does not satisfy condition (10) then problem (1)-(3) in a simple way may be reduce to a problem right hand side of which satisfies condition (10).

REFERENCES

1. Bitsadze A.V., Samarskii A.A. Some elementary generalizations of linear elliptic boundary value problems. (Russian) *Dokl. Akad. Nauk SSSR,* **185**, 4 (1969), 739-740.

2. Gordeziani, D.G. A method for solving the Bitsadze-Samarskii boundary value problem. (Russian) *Abstr. Sem. Inst. Appl. Math. TSU*, **2** (1970), 39-41.

3. Jangveladze T.A. On One Iterative Method of Solution of Bitsadze-Samarskii Boundary Value Problem. (Georgian) Manuscript, Faculty of Applied Mathematics and Cybernetics of Tbilisi State University, *Tbilisi,* 1975.

4. Gordeziani D.G. Methods for Solving a Class of Nonlocal Boundary Value Problems. (Russian) Tbilis. Gos. Univ., Inst. Prikl. Mat., *Tbilisi,* 1981.

5. Il'in V.A., Moiseev E.I. A nonlocal boundary value problem of the first kind for the Sturm-Liouville operator in differential and difference interpretations. (Russian) *Differ. Uravn.,* **23**, 7 (1987), 1198-1207.

6. Sapagovas M.P., Chegis R.Yu. On some boundary value problems with a nonlocal condition. (Russian) *Differ. Uravn.,* **23**, 7 (1987), 1268-1274.

7. Gushchin A.K., Mikhailov V.P. Continuity of solutions of a class of nonlocal problems for an elliptic equation. (Russian) *Mat. Sb.,* **133(175)**, 2 (1995), 37-58.

8. Kiguradze Z.V. Domain decomposition and parallel algorithm for Bitsadze-Samarskii boundary value problem. *Rep. Enlarged Sess. Sem. I.Vekua Appl. Math.,* **10**, 1 (1995), 49-51.

9. Skubachevskii A.L. Elliptic Functional Differential Equations and Applications. *Basel: Birkh¨Aauser,* 1997.

10. Lobjanidze G. Remark on the variational formulation of Bitsadze-Samarskii nonlocal problem. *Rep. Enlarged Sess. Sem. I.Vekua Appl. Math.,* **16**, 1-3 (2001), 102-103.

11. Lobjanidze G. On the variational formulation of Bitsadze-Samarskii problem for the equation $-\Delta u + \lambda u = f$. Rep. *Enlarged Sess. Sem. I.Vekua Appl. Math.,* **18**, 1 (2003), 39-40.

12. Lobjanidze G. On one variational interpretation of Bitsadze-Samarskii non-local boundary value problem for the biharmonic equation. *Rep. Enlarged Sess. Sem. I.Vekua Appl. Math.,* **20**, 3 (2005), 103-106.

13. Jangveladze T.A., Kiguradze Z.V. Domain decomposition method for Bitsadze-Samarskii boundary value problem. (Russian) *Trudy Tbiliss. Univ. Mat. Mekh. Astronom.,* **354**, 34-35 (2005), 225-236.

14. Jangveladze T. Methods of Approximate Solution of Ordinary Differential Equations. (Georgian) University Press, Tbilisi, 2005.

15. Jangveladze T., Lobjanidze G., Kiguradze Z. On variational formulation and decomposition methods for Bitsadze-Samarskii nonlocal boundary value problem for two-dimensional second order elliptic equations. *Proc. I.Vekua Inst. Appl. Math.,* **56-57** (2006-2007), 66-77.

16. Jangveladze T., Lobjanidze G. Variational formulation of one nonlocal boundary problem. *Rep. Enlarged Sess. Sem. I.Vekua Appl. Math.,* **23** (2009), 46-49.

17. Jangveladze T., Lobjanidze G. On a variational statement of a nonlocal boundary value problem for a forth-order ordinary differential equation. (Russian) *Differ. Uravn.,* **45**, 3 (2009), 325-333. English translation: *Differ. Equ.,* **45**, 3 (2009), 335-343.

18. Rektorys K. Variational Methods in Mathematics, Science and Engineering. *Dordrecht (Netherlands): Reidel,* 1983. Translated under the title Variatsionnye Metody v Matematicheskoi Fizike i Tekhnike, *Moscow: Mir,* 1985.

Received 13.05.2010; revised 16.09.2010; accepted 25.10.2010.

Authors' address:

G. Lobjanidze
Caucasus University
Kostava Av. 77, 0175
Georgia
E-mail: mariamo.lobzhanidze@yahoo.com