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# STATEMENT AND EFFECTIVE SOLUTION OF SOME NONCLASSICAL PROBLEMS OF THERMOELASTICITY FOR A RECTANGULAR PARALLELEPIPED

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**Abstract**. The static thermoelastic equilibrium of an isotropic homogenous rectangular parallelepiped is considered. Boundary conditions of antisymmetry or symmetry are given on the lateral faces of the parallelepiped, the upper and lower faces are free from stresses. Thermal disturbance is given on the lower face. The problem consists in giving a temperature on the upper face of the parallelepiped so, that on some plane inside the body which is parallel to the bases the normal displacement would take a given value. The stated nonclassical problem is solved analytically by the method of separetion of variables.

**Keywords and phrases**: thermoelasticity, nonclassical problem, rectangular parallelepiped.

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**Introduction.** In the elasticity theory there are quite a number of problems which could be called nonclassical because the boundary conditions on a part of the boundary surface are either overdefined or underdefined [1], [2], or the conditions on the boundary are related to the conditions in the interior of a body (the so-called nonlocal problems) [3]-[5].

In the present paper, we formulate and solve effectively by the method of separation of variables the following nonclassical problems of thermoelasticity for the rectangular parallelepiped.

In the Cartesian system of $x$, $y$, $z$ coordinates, we consider the thermoelastic equilibrium of a homogeneous isotropic body occupying the domain $\Omega = \{0 < x < x_1, 0 < y < y_1, 0 < z < z_1\}$. The boundary conditions of symmetry or antisymmetry [6] are given on the lateral faces of $\Omega$; the faces $z = 0$ and $z = z_1$ are assumed to be stress-free; a temperature disturbance is given on the face $z = 0$.

The problem consists in the following: on the face $z = z_1$, define a temperature such that the function of normal displacements $w(x, y, z)$ of points of some inner surface $z = z_2$ of the body would take a prescribed value. Naturally, after solving the stated problem we can easily find the stress-strained state of the considered body.

**1. Statement of problems.** We consider a given homogeneous isotropic elastic body, which in the Cartesian system of $x$, $y$, $z$ coordinates, occupies the domain $\Omega = \{0 < x < x_1, 0 < y < y_1, 0 < z < z_1\}$, where $x_1$, $y_1$, $z_1$ are constants. The considered body is in the state of stationary thermoelastic equilibrium. On the lateral faces of the domain $\Omega$ the following boundary conditions are given [6]:

$$\left.\begin{array}{l} \text{for } x = x_j\text{: a) the conditions of antisymmetry } \sigma_{xx} = 0,\ v = 0,\ w = 0,\ T = 0 \\ \qquad\qquad\text{or} \\ \qquad\qquad\text{b) the conditions of symmetry } u = 0,\ \sigma_{xy} = 0,\ \sigma_{xz} = 0,\ \partial_x T = 0; \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{l}\text{for } y = y_j\text{: a) the conditions of antisymmetry } \sigma_{yy} = 0,\ u = 0,\ w = 0,\ T = 0 \\ \qquad\qquad\text{or} \\ \qquad\text{b) the conditions of symmetry } v = 0,\ \sigma_{yx} = 0,\ \sigma_{yz} = 0,\ \partial_y T = 0.\end{array}\right\} \qquad (2)$$

On the upper and lower faces of the parallelepiped the following conditions are given:

$$\text{for } z = z_j\text{:} \qquad \sigma_{zx} = 0, \quad \sigma_{zy} = 0, \quad \sigma_{zz} = 0; \qquad (3)$$

$$\left.\begin{array}{ll}\text{for } z = 0\text{:} & \text{a) } T = \tau(x, y) \quad \text{or} \quad \text{b) } \partial_z T = \widetilde{\tau}(x, y) \quad \text{or} \\ & \text{c) } \partial_z T + \Theta T = \widetilde{\tau}(x, y),\end{array}\right\} \qquad (4)$$

where $j = 0, 1$, $x_0 = y_0 = z_0 = 0$; $u$, $v$, $w$ are the components of the displacement vector $\vec{U}$ along the $x$, $y$, $z$ components, respectively; $T$ is the temperature change in the elastic body which satisfies the equation

$$\Delta T = 0, \qquad (5)$$

where $\Delta \equiv \partial_{xx} + \partial_{yy} + \partial_{zz}$; $\partial_x \equiv \frac{\partial}{\partial x}$, $\partial_y \equiv \frac{\partial}{\partial y}$, $\partial_z \equiv \frac{\partial}{\partial z}$; $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{zz}$ are normal stresses; $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, $\sigma_{yz} = \sigma_{zy}$ are tangential stresses; $\Theta$ is the given constant; $\tau(x, y)$, $\widetilde{\tau}(x, y)$ are the given analytic functions of the variables $x$ and $y$ in the domain $\overline{\omega} = \{0 \le x \le x_1,\ 0 \le y \le y_1\}$.

The problem consists in that on the face $z = z_1$ we must define a change of the temperature $T$ in such a manner that on the inner surface of the body $z = z_2$ $(0 < z_2 < z_1)$ the condition

$$w(x, y, z_2) = g(x, y) \qquad (6)$$

would be fulfilled, where $g(x, y)$ is the given analytic function of the variables $x$ and $y$ in the domain $\overline{\omega}$.

In the absence of mass forces the thermoelastic equilibrium of an isotropic homogeneous elastic body is described, as is known [7], by the following differential equation

$$\operatorname{grad}\left[2(1-\nu)\operatorname{div}\vec{U} - 2(1+\nu)kT\right] - (1-2\nu)\operatorname{rot}\operatorname{rot}\vec{U} = 0, \qquad (7)$$

where $E$ is Young's modulus; $\nu$ is Poisson's ratio; $k$ if the coefficient of linear thermal expansion.

**2. Solution of the stated problems.** The solution of the stated problems is carried out by the method of separation of variables taking into account the results of the paper [7], where a general solution of the system of equations (7) is represented through three arbitrary harmonic functions and the function $\widetilde{T}$ which is also a solution of the Laplace equation and is related to a change of the temperature $T$ by

$$T = \partial_{zz}\widetilde{T}. \qquad (8)$$

In the same paper [7], it is shown that in the case of the boundary conditions (1)–(3) all the above-mentioned harmonic functions, except for $\widetilde{T}$, are equal to zero, while displacements and stresses are expressed through the function $\widetilde{T}$ as follows:

$$w = k(1+\nu)\partial_z \widetilde{T}; \tag{9}$$

$$u = -k(1+\nu)\partial_x \widetilde{T}, \quad v = -k(1+\nu)\partial_y \widetilde{T}; \tag{10}$$

$$\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0; \tag{11}$$

$$\sigma_{xx} = -\frac{E}{1+\nu}\,\partial_y v, \quad \sigma_{yy} = -\frac{E}{1+\nu}\,\partial_x u, \quad \sigma_{yx} = \frac{E}{2(1+\nu)}\,(\partial_y u + \partial_x v). \tag{12}$$

Let us first construct the solution of the problem (7), (5), (1a), (2a), (3), (4a). It is assumed that the antisymmetry conditions are given on all four faces of the parallelepiped.

Using the method of separation of variables, by virtue of the relation (8) and the boundary conditions (1a) and (2a) we can represent the function $\widetilde{T}$ as

$$\widetilde{T} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\gamma_{mn}^2}\left(A_{mn}e^{-\gamma_{mn}z} + B_{mn}e^{\gamma_{mn}z}\right)\sin\frac{\pi m x}{x_1}\,\sin\frac{\pi n y}{y_1}\,, \tag{13}$$

where $\gamma_{mn} = \sqrt{\left(\frac{\pi m}{x_1}\right)^2 + \left(\frac{\pi n}{y_1}\right)^2}$ ; $A_{mn}$ and $B_{mn}$ are the constants depending on $m$ and $n$.

Since the given analytic functions $\tau(x,y)$ and $g(x,y)$ satisfy the consistency condition, we expand them into trigonometric sine-series by the formulas

$$\tau(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tau_{mn}\sin\frac{\pi m x}{x_1}\,\sin\frac{\pi n y}{y_1}\,, \tag{14}$$

$$g(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}\sin\frac{\pi m x}{x_1}\,\sin\frac{\pi n y}{y_1}\,. \tag{15}$$

It will be assumed that the Fourier coefficients $\tau_{mn}$ and $g_{mn}$ satisfy the following conditions

$$\tau_{mn} = O\left(\frac{1}{e^{\gamma_{mn}(z_1-z_2)}}\right), \quad g_{mn} = O\left(\frac{1}{e^{\gamma_{mn}z_1}}\right). \tag{16}$$

From (13) with (8) taken into account, for the change of the temperature $T$ we obtain the expression

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}e^{-\gamma_{mn}z} + B_{mn}e^{\gamma_{mn}z}\right)\sin\frac{\pi m x}{x_1}\,\sin\frac{\pi n y}{y_1}\,. \tag{17}$$

Substituting the representation (13) in (9), for the normal displacement $w$ we have

$$w = k(1+\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\gamma_{mn}}\left(-A_{mn}e^{-\gamma_{mn}z} + B_{mn}e^{\gamma_{mn}z}\right)\sin\frac{\pi m x}{x_1}\,\sin\frac{\pi n y}{y_1}\,. \tag{18}$$

Substituting the series (17) and (14) in the boundary condition (4a) and the series (18) and (15) in the condition (6) and equating the coefficients of identical trigonometric functions to each other, for the unknown coefficients $A_{mn}$ and $B_{mn}$ we obtain the following system of equations

$$\begin{cases} A_{mn} + B_{mn} = \tau_{mn}, \\ -A_{mn}e^{-\gamma_{mn}z_2} + B_{mn}e^{\gamma_{mn}z_2} = \frac{\gamma_{mn}}{k(1+\nu)}\, w_{mn}, \quad m,n \in N. \end{cases} \tag{19}$$

Defining the coefficients $A_{mn}$ and $B_{mn}$ from the system (19) and substituting them in the formula (17), for the change of the temperature $T$ we obtain the expression

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\operatorname{ch}(\gamma_{mn}z_2)} \left\{ \operatorname{ch}(\gamma_{mn}(z-z_2))\tau_{mn} + \frac{\gamma_{mn}}{k(1+\nu)}\, \operatorname{sh}(\gamma_{mn}z)g_{mn} \right\} \times \sin\frac{\pi m x}{x_1}\, \sin\frac{\pi n y}{y_1}\,. \tag{20}$$

It can be easily shown that if the coefficients $\tau_{mn}$ and $g_{mn}$ satisfy the conditions (16), then the obtained series (20) converges absolutely and uniformly in the domain $\overline{\Omega}$. Moreover, the obtained function $T$ will be an analytic function of the variables $x$, $y$, $z$ in the domain $\overline{\Omega}$.

Replacing $z$ by $z_1$ in the formula (20), we obtained the desired value of the function $T$ on the face $z = z_1$. It is the unique solution of the considered problem. It is not difficult to prove that the obtained value depends continuously on the initial data provided that the Fourier coefficients of the functions $\tau^*(x,y)$ and $g^*(x,y)$, which are disturbances of the functions $\tau(x,y)$ and $g(x,y)$, also satisfy the conditions (16).

Knowing the function $\widetilde{T}$, by means of the formulas (9)–(16) we can easily define displacements and stresses in the considered body.

Let us consider the concrete example where the given functions $\tau(x,y)$ and $g(x,y)$ are written as

$$\tau(x,y) = \tau_{22}\sin\frac{2\pi x}{x_1}\,\sin\frac{2\pi y}{y_1}\,, \quad g(x,y) = w_{11}\sin\frac{\pi x}{x_1}\,\sin\frac{\pi y}{y_1}\,.$$

In that case, the formula (20) implies the following elementary expression for the function $T$,

$$T = \frac{\gamma_{11}w_{11}\operatorname{sh}(\gamma_{11}z)}{k(1+\nu)\operatorname{ch}(\gamma_{11}z_2)}\,\sin\frac{\pi x}{x_1}\,\sin\frac{\pi y}{y_1} + \frac{\tau_{22}\operatorname{ch}(\gamma_{22}(z-z_2))}{\operatorname{ch}(\gamma_{22}z_2)}\,\sin\frac{2\pi x}{x_1}\,\sin\frac{2\pi y}{y_1}\,.$$

Let us also write the solution of the problem when the antisymmetry conditions (1a) and (2a) are given on the parallelepiped faces $x = 0$ and $y = 0$, and the symmetry conditions (1b) and (2b) are given on the faces $x = x_1$ and $y = y_1$. It has the form

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\operatorname{ch}(\eta_{mn}z_2)} \left\{ \operatorname{ch}(\eta_{mn}(z-z_2))\tau_{mn} + \frac{\gamma_{mn}}{k(1+\nu)}\, \operatorname{sh}(\eta_{mn}z)g_{mn} \right\} \times \sin\frac{(2m-1)\pi x}{2x_1}\, \sin\frac{(2n-1)\pi y}{2y_1}\,.$$

The solutions of our other stated nonclassical problems are solved in an absolutely analogous manner.

To conclude, we indicate that by taking into consideration the results of [7] the corresponding nonclassical problems of thermoelasticity can be formulated and solved in a generalized cylindrical system of coordinates, assuming that an elastic body may transversally isotropic with isotropy plane $z = const$. We are convinced that the solution of all these problems may have a practical application.

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