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A GENERALIZED NONLOCAL BOUNDARY VALUE PROBLEM IN A SPACE $L_p(\partial\Omega)$, $1<p<\infty$

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Abstract. The generalized nonlocal boundary value problem for Laplace equation is studied.

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The nonlocal boundary value problem is considered in [1-4] when the closed surface S belongs to the domain Ω ($S\subset\Omega$). In the present paper we consider the case where the intersection $\partial\Omega\cap S$ contains a finite number of points, $z_i\in\partial\Omega\cap S$, $i=1,2,\dots,N$. The compactness of the operator K, $K\varphi(x)=v_\varphi(Z(x)), x\in\partial\Omega, \varphi\in C(\partial\Omega)$ [4] cannot be proved in the space $C(\partial\Omega)$. But the compactness of K can be proved in the space $L_p(\partial\Omega)$, $1<p<\infty$. Assume that $\Omega\subset R^3$ is a simply connected bounded domain from the class $C^{(2,\alpha)}$, the closed surface $S\subset\Omega$, $S\in C^{(2,\alpha)}$. Denote $\sigma_k=\{x:|x-z_k|<\delta\}\cap\partial\Omega$, $\gamma_k=\{\zeta:|\zeta-z_k|<\delta\}\cap S$. Let ν_k be the normal for S and $\partial\Omega$ at the point z_k. If δ is a sufficiently small number, then $m(\gamma_k)<\frac{\varepsilon}{N}$, $m(\sigma_k)<\frac{\varepsilon}{N}$, where m is a two-dimensional Lebesgue measure. Let p_{1k} be the projection of σ_k on the tangential plane P_k at the point z_k, and p_{2k} be the projection of σ_k on P_k. Denote $e'_k=p_{1k}\gamma_k\cap p_{2k}\sigma_k$, $e_k=|z_k-\eta|<r_k$, $r_k=\inf|z_k-\eta|$, $\eta\in\partial e'_k$. Let us define a diffeomorphism in the neighborhood of z_k $\zeta=z_k(x)$ if $p_{1k}(\zeta)=p_{2k}(x)\in e_k$, $\nu_k\perp p_k$.

Let us assume that there exists a $C^{(2,\alpha)}$-diffeomorphism $\zeta=z(x)$ from $\partial\Omega$ on S that satisfies the condition $z(x)=z_k(x)$ if $p_{2k}(x)\in e_k$.

Assume that the boundary function $f\in L_p(\partial\Omega)$, $1<p<\infty$. One has fined a function $\varphi\in L_p(\partial\Omega)$, satisfying the boundary condition

$$\varphi-K\varphi=f, \tag{1}$$

where

$$K\varphi(x)=v_\varphi(z(x))=-\int\limits_{\partial\Omega}\frac{\partial G(Z(x),y)}{\partial\nu_y}\varphi(y)dS_y$$

Denote

$$I_k(\zeta)=\begin{cases}1, & p_{1k}\zeta\in e_k \quad (\zeta\in\gamma_k),\\ 0, & \zeta\in S, \quad p_{1k}\zeta\notin e_k.\end{cases}$$

Thus we obtain a finite-dimensional space

$$\tau(\zeta)=\sum_{k=1}^{N}\alpha_k I_k(\zeta), \quad -\infty<\alpha_k<\infty, \quad k=1,2,\dots,N.$$

We define the sweep-out operator for I_k from S on $\partial\Omega$:

$$TI_k(y) = -\int_{S_k} \frac{\partial G(\zeta, y)}{\partial \nu_y} Dz_k^{-1}(\zeta)\, dS_k, \quad y \in \partial\Omega, \ \ |Dz_k^{-1}| \le c_1,$$
$$S_k = \{\zeta : p_{1k}\zeta \in e_k, \ \ \zeta \in \gamma_k\}.$$

Let

$$\tau_1(\zeta) = \sum_{k=1}^{N} I_k(\zeta), \quad \|\tau_1\|_{L_1} \le \sum m(\gamma) < \varepsilon, \tag{2}$$

$$T\tau_1(y) = \sum_{k=1}^{N} TI_k(y) \in C(\partial\Omega), \quad \|T\tau_1\|_{C(\partial\Omega)} \le c_2\|\tau 1\|_{L_1} \le c_2 \cdot \varepsilon.$$

Let $\varphi_n \in L_p(\partial\Omega)$, $1 < p < \infty$, $\|\varphi_n\|_p \le M$, $n = 1, 2, \ldots$

$$K\varphi_n(x) = -\int_{\partial\Omega} \frac{\partial G(z(x), y)}{\partial \nu_y} \varphi_n(y) dS_y, \ \ x \in \sigma_k, \ \ p_{2k}x \in e_k, \ \ E_k = \{x : p_{2k}(x) \in e_k\},$$

$$\int_{\bigcup\limits_{k=1}^{N} E_k} |K\varphi_n(x)|^p dS_x = \int_{\bigcup\limits_{k=1}^{N} E_k} \int_{\partial\Omega} \frac{\partial G(z(x), y)}{\partial \nu_y} |\varphi_n(y)|^p dS_y\, dS_y \le \left(\bigcup_{k=1}^{N} = E \right)$$
$$\le \int_{\partial\Omega} \int_{E} \frac{\partial G(\zeta, y)}{\partial \nu_y} Dz^{-1}(\zeta)\, dS_\zeta |\varphi_n(y)|^p dS_y \le c_3 \cdot \varepsilon \sup_n \int_{\partial\Omega} |\varphi_n(y)|^p dS_y.$$

Therefore

$$\left(\int_E |K\varphi_n(x)|^p dS_x \right)^{\frac{1}{p}} \le (c_3 \cdot \varepsilon)^{\frac{1}{p}} \cdot M, \quad F = \bigcup_{k=1}^{n} F_k, \quad F_k = Z(E_k).$$

Thus there exists a subsequence $K\varphi_j$, $j = 1, 2, \ldots$, for which we obtain

$$\left| \frac{\partial G(\zeta, y)}{\partial \nu_y} \right| \le \frac{c_4}{|\zeta - y|^2} \quad \zeta \in S - \bigcup_{k=1}^{N} p_{1k}^{-1}(e_k), \quad y \in \partial\Omega,$$
$$\lim_{j\to\infty} K\varphi_j(x) = K\varphi(x), \quad z(x) \in \Omega.$$

Hence it follows that

$$\sup_j |K\varphi_j(x)| \le L_1 < \infty, \quad x \in \partial\Omega - \bigcup_{k=1}^{N} p_{2k}^{-1} e_k.$$

Thus K is a compact operator from $L_p(\partial\Omega)$ into $L_p(\partial\Omega)$.

Let us consider the conjugate operator K^* and the corresponding homogeneous equation

$$g - K^* g = 0, \quad g \in L_q(\partial\Omega), \quad q = \frac{p}{p-1}\,.$$

Let $g_i (i = 1, 2, \ldots, k)$ solutions for homogeneous equation. Finally let us define the space B_1 of boundary functions:

$$B_1 = \left\{ f : f \in L_p(\partial\Omega),\ \int_{\partial\Omega} f(x) g_i(x) dS_x = 0,\ i = 1, 2, \ldots, k \right\}.$$

Theorem. *The nonlocal boundary problem (1) in space $L_p(\partial\Omega)$ is solvable if and only if $f \in B_1$.*

R E F E R E N C E S

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