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ADDITIVE MODELS FOR ONE NONLINEAR DIFFUSION SYSTEM

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Abstract. Nonlinear diffusion parabolic model based on Maxwell's system is considered. Joule's rule and thermal conductivity are taking into account. Semi-discrete averaged additive models are studied for this system.

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Process of penetration of magnetic field into a substance is accompanied with thermal phenomena, which essentially changes process of diffusion and complicates corresponding system of Maxwell's equations [1]. Mentioned system, taking into account Joule's rule and thermal conductivity, has the following form:

$$\frac{\partial \theta}{\partial t} = \nu_m (rot\, H)^2 + div(\kappa\, grad\, \theta), \qquad \frac{\partial H}{\partial t} = -rot\ (\nu_m rot\, H), \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ - temperature, ν_m and κ are characteristics coefficients of the substance. As a rule these coefficients are functions of argument θ.

The questions of existence, uniqueness, regularity, asymptotic behavior of the solutions and numerical resolution of the initial-boundary value problems to the (1) type models are discussed in many works (see, for example, [2]-[13] and references therein).

Beside of essential nonlinearity, complexities of the mentioned system (1) is caused by its multi-dimensionality. This circumstance is complicating to get numerical results for concrete real problems. Naturally arises the possibility of reduction to suitable one-dimensional models. Complex nonlinearity dictates also to split along the physical process and investigate basic model by them. In particular, it is logical to split system (1) into following two models (see, for example, [3], [13]):

$$\frac{\partial \tilde{H}}{\partial t} = -rot\, (\nu_m(\tilde{\theta}) rot\, \tilde{H}), \qquad \frac{\partial \tilde{\theta}}{\partial t} = \nu_m(\tilde{\theta})(rot\, \tilde{H})^2 \tag{2}$$

and

$$\frac{\partial \tilde{\tilde{\theta}}}{\partial t} = div\, (\kappa(\tilde{\tilde{\theta}}) grad\, \tilde{\tilde{\theta}}). \tag{3}$$

In (2) Joule's rule, while in (3) process of thermal conductivity are considered.

Different type of splitting-up schemes are constructed and investigated for many models of mathematical physics (see, for example, [14]-[18] and references therein).

Note that, system (2) can be reduced to integro-differential form. This reduction at first was made in the work [19]. Many works were followed after publication of this paper (see, for example, [20]-[28] and references therein). The questions of existence, uniqueness, asymptotic behavior of the solutions and numerical resolution of some kind of initial-boundary value problems for this type integro-differential models are studied in these works.

Investigation of splitting-up schemes along the physical processes for one-dimensional analog of system (1) is the natural beginning of studding this issue. In this direction the investigations was made in the works [3] and [13]. In the paper [3] the initial-boundary value problem with the first kind boundary conditions for the temperature are considered.

The aim of this note is to construct and study additive analogues bases on models (2) and (3) for one-dimensional analog of system (1) with one-component magnetic field with the second kind boundary conditions for the temperature. At the end of this note we also state additive model for multi-dimensional system (1).

In the domain $Q = \Omega \times (0, T)$ let us consider following problem for one-dimensional analog of system (1):

$$\begin{gathered}\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(V^\alpha \frac{\partial U}{\partial x}\right), \qquad \frac{\partial V}{\partial t} = V^\alpha \left(\frac{\partial U}{\partial x}\right)^2 + \frac{\partial^2 V}{\partial x^2}, \\ U(x,t) = \frac{\partial V(x,t)}{\partial x} = 0, \ \ (x,t) \in \partial\Omega \times (0,T), \\ U(x,0) = U_0(x), \ \ V(x,0) = V_0(x) \geq Const > 0,\end{gathered} \tag{4}$$

where $-1/2 \leq \alpha \leq 1/2$; U_0, V_0 are known functions defined on $\Omega = [0,1]$, T is the fixed positive number.

If we denote $V^{1/2} = W$, $2\alpha = \gamma$, then problem (4) can be rewritten in the following equivalent form:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(W^\gamma \frac{\partial U}{\partial x}\right), \qquad \frac{\partial W}{\partial t} = \frac{1}{2}W^{\gamma-1}\left(\frac{\partial U}{\partial x}\right)^2 + \frac{\partial^2 W}{\partial x^2} + \frac{1}{W}\left(\frac{\partial W}{\partial x}\right)^2, \tag{5}$$

$$\begin{gathered}U(x,t) = \frac{\partial W(x,t)}{\partial x} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ U(x,0) = U_0(x), \quad W(x,0) = W_0(x) = V_0^{1/2}(x),\end{gathered}$$

where $-1 \leq \gamma \leq 1$.

Let us introduce the uniform grid $\omega_\tau = \{t_j = j\tau, \ \ j = 0, 1, ..., N\}$ on $[0, T]$. Using the notations:

$$\begin{gathered}y_t = \frac{y^{j+1} - y^j}{\tau}, \quad \eta_1 + \eta_2 = 1, \quad \eta_1 > 0, \quad \eta_2 > 0, \\ y = \eta_1 y_1 + \eta_2 y_2, \quad y_{1t} = \frac{y_1^{j+1} - y^j}{\tau}, \quad y_{2t} = \frac{y_2^{j+1} - y^j}{\tau},\end{gathered}$$

let us correspond to the initial-boundary value problem (5) following additive averaged semi-discrete scheme:

$$\begin{gathered}u_{1t} = \frac{d}{dx}\left(w_1^\gamma \frac{du_1}{dx}\right), \quad \eta_1 w_{1t} = \frac{1}{2}w_1^{\gamma-1}\left(\frac{du_1}{dx}\right)^2, \\ u_{2t} = \frac{d}{dx}\left(w_2^\gamma \frac{du_2}{dx}\right), \quad \eta_2 w_{2t} = \frac{d^2 w_2}{dx^2} + \frac{1}{w_2}\left(\frac{dw_2}{dx}\right)^2, \\ u_1^0 = u_2^0 = U_0, \quad \frac{dw_1^0}{dx} = \frac{dw_2^0}{dx} = W_0.\end{gathered} \tag{6}$$

The following statement takes place.

Theorem. *If problem (5) has a sufficiently smooth solution, then the solution of the scheme (6) converges to the solution of problem (5) as* $\tau \to 0$ *and the following estimate is true*

$$\left\|U(t_j) - u^j\right\| + \left\|W(t_j) + w^j\right\| = O(\tau^{1/2}).$$

Here $\|\cdot\|$ is an usual norm of the space $L_2(0,1)$.

Note also that the result analogical to above theorem is true for the following semi-discrete additive model corresponding again to the problem (5):

$$\begin{gathered}u_t=\frac{d}{dx}\left(\left(\eta_1 w_1^\gamma+\eta_2 w_2^\gamma\right)\frac{du}{dx}\right),\quad \eta_1 w_{1t}=\frac{1}{2}w_1^{\gamma-1}\left(\frac{du}{dx}\right)^2,\\ \eta_2 w_{2t}=\frac{d^2 w_2}{dx^2}+\frac{1}{w_2}\left(\frac{dw_2}{dx}\right)^2,\end{gathered}\tag{7}$$

with suitable initial and boundary conditions.

At last we note that we can also construct additive models analogical to (6) and (7) for the system (1). For example, after rewriting system (1) in form analogical to (5):

$$\frac{\partial W}{\partial t}=\frac{1}{2}W^{\gamma-1}(rot\,H)^2+\frac{1}{2W}div(\kappa\,grad\,W),\quad \frac{\partial H}{\partial t}=-rot(W^\gamma rot\,H).\tag{8}$$

The averaged semi-discrete scheme for (8) has the following form:

$$\eta_1 w_{1t}=\frac{1}{2}w_1^{\gamma-1}(rot\ u)^2,\quad u_t=-rot(w_1^\gamma rot\ u),\quad \eta_2 w_{2t}=\frac{1}{2w_2}div(\kappa\,grad\ w_2).\tag{9}$$

It is proved that the statement analogical to above formulated theorem is true for (9) model as well.

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