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UNIQUENESS THEOREMS IN THE THERMOELASTICITY THEORY OF ANISOTROPIC BODIES

Ivanidze M., Ivanidze D.

Abstract. In the paper we prove the uniqueness theorems for the interior and exterior Dirichlet and Neumann type boundary value problems of thermoelastostatics for general anisotropic elastic solids.

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We prove the uniqueness of solutions to the static exterior boundary value problems of the theory of thermoelasticity of anisotropic bodies. The basic equations read as [1], [2]:

$$c_{kjpq}\partial_j\partial_q u_p(x) - \beta_{kj}\partial_j\vartheta(x) = \Phi_k, \quad k = 1, 2, 3, \tag{1}$$

$$\lambda_{pq}\partial_p\partial_q\vartheta(x) = \Phi_4, \tag{2}$$

where $c_{kjpq} = c_{pqkj} = c_{jkpq}$ are the elastic constants, $\lambda_{pq} = \lambda_{qp}$ are the heat conductivity coefficients, $\beta_{pq} = \beta_{qp}$ are the thermal strain constants, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, ϑ is the temperature function, Φ_k are mass forces, while Φ_4 is a heat source function.

Let us introduce the natation: $U = (u_1, u_2, u_3, \vartheta)^\top = (u, \vartheta)^\top$, $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$,

$$C(\partial) = [C_{kp}(\partial)]_{3\times3}, \quad C_{kp}(\partial) = c_{kjpq}\partial_j\partial_q, \quad \Lambda(\partial) = \lambda_{pq}\partial_p\partial_q,$$

$$A(\partial) := [A_{kj}(\partial)]_{4\times4} = \begin{bmatrix} C(\partial) & [-\beta_{kj}\partial_j]_{3\times1} \\ 0 & \Lambda(\partial) \end{bmatrix}_{4\times4}, \quad \partial = (\partial_1, \partial_2, \partial_3), \quad \partial_j = \frac{\partial}{\partial x_j}.$$

Then equations (1) and (2) can be rewritten then in matrix form

$$A(\partial)U(x) = \Phi. \tag{3}$$

Further let $\Omega^+ \subset \mathbb{R}^3$ is a bounded domain and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$, $S = \partial\Omega^\pm \in C^{1,\alpha}$, and consider the following exterior Dirichlet problem:
Find a vector $U = (u, \vartheta)^\top \in [C^{2,\alpha}(\Omega^-)]^4 \cap [C^{1,\alpha}(\overline{\Omega^-})]^4$ with $0 < \alpha \le 1$ satisfying the differential equation (3) and the boundary condition:

$$[U(x)]^- = \varphi(x), \quad x \in S, \tag{4}$$

where $\Phi \in [C^{0,\alpha}_{comp}(\Omega^-)]^4$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^\top \in [C^{1,\alpha}(S)]^4$ are given vectors.

It is clear that the Dirichlet type boundary value problem for the temperature function $\vartheta(x)$ is separated:

$$A_{44}(\partial)\,\vartheta(x) = \lambda_{pq}\partial_p\partial_q\vartheta(x) = \Phi_4(x),\ \ x\in\Omega^-, \tag{5}$$
$$[\vartheta(x)]^- = \varphi_4(x),\ \ \ x\in S. \tag{6}$$

If we assume that

$$\vartheta(x) = \mathcal{O}(|x|^{-1})\ \text{ as }\ |x|\to\infty,$$

then for arbitrary $\Phi_4 \in C^{0,\alpha}_{comp}(\Omega^-)$ and $\varphi_4 \in C^{1,\alpha}(S)$ the BVP (5)-(6) is uniquely solvable in the space $C^{2,\alpha}(\Omega^-)\cap C^{1,\alpha}(\overline{\Omega^-})$. Moreover, there holds the representation in Ω^- [2], [3]:

$$\vartheta(x) = \int_S \Gamma^{(0)}_{44}(x-y)[\partial_{n(y)}\vartheta(y)]^- ds_y - \int_S \partial_{n(y)}\Gamma^{(0)}_{44}(x-y)[\vartheta(y)]^- ds_y + \int_{\Omega^-}\Gamma^{(0)}_{44}(x-y)\Phi_4(y)dy,$$

where $\partial_{n(y)} = \lambda_{pq}n_q(y)\partial_p$ denotes the co-normal derivative and

$$\Gamma_{44}(x) = \frac{\alpha_0}{(Dx\,,\,x)^{\frac12}},\ \ \alpha_0 = -\frac{1}{4\pi\,[\det A]^{\frac12}},\ A=[\lambda_{pq}]_{3\times3},\ D=A^{-1}=[d_{kj}]_{3\times3},$$

is a fundamental solution of the operator $A_{44}(\partial)$. Since Φ_4 has a compact support, it follows that we have the following asymptotic formulas

$$\vartheta(x) = \frac{\theta_0}{(Dx,x)^{1/2}} + \mathcal{O}(|x|^{-2}),\quad \partial_j\,\vartheta(x) = -\frac{\theta_0\,d_{jm}\,x_m}{(Dx,x)^{3/2}} + \mathcal{O}(|x|^{-3})\text{ as }|x|\to\infty,$$

with some constant θ_0.

Thus, if we assume that the the temperature function is known, from (3) and (4) we arrive at the exterior Dirichlet BVP for the displacement vector u

$$C(\partial)u(x) = \widetilde{\Psi} + \widetilde{\Phi},\quad x\in\Omega^-, \tag{7}$$
$$[u(x)]^- = \widetilde{\varphi}(x),\quad x\in S, \tag{8}$$

where $\widetilde{\varphi} = (\varphi_1,\varphi_2,\varphi_3)^\top \in [C^{1,\alpha}(S)]^3$,

$$\widetilde{\Phi} = (\Phi_1,\Phi_2,\Phi_3)^\top \in [C^{0,\alpha}_{comp}(\Omega^-)]^3,\ \ \widetilde{\Psi} = (\beta_{1j}\partial_j\vartheta,\,\beta_{2j}\partial_j\vartheta,\,\beta_{3j}\partial_j\vartheta\,)^\top \in [C^{0,\alpha}(\overline{\Omega^-})]^3.$$

Note that $\widetilde{\Psi}$ has not compact support and $\widetilde{\Psi} = \widetilde{Q}(x) + \theta_0\widetilde{P}(x)$ with

$$\widetilde{\Psi}(x) = -\frac{1}{(Dx,x)^{3/2}}\,(\,\beta_{1j}\,d_{j\ell}\,x_\ell\,,\,\beta_{2j}\,d_{j\ell}\,x_\ell\,,\,\beta_{3j}\,d_{j\ell}\,x_\ell\,)^\top,\ \ \widetilde{Q}(x) = \mathcal{O}(|x|^{-3}),\ \ |x|\to\infty.$$

Since $\widetilde{\Psi}(x) = \mathcal{O}(|x|^{-2})$ we can not assume that u vanishes at infinity.

Our goal is to establish sufficient conditions insuring the uniqueness of solutions to the problem (7)-(8) in the space of bounded vectors.

To this end we need several auxiliary propositions [4].

Lemma 1. *Let $u = (u_1, u_2, u_3)^\top$ be a bounded solution to the homogeneous differential equation*

$$C(\partial)u(x) = 0, \quad x \in \Omega^-.$$

Then

$$u(x) = c + \mathcal{O}(|x|^{-1}) \ \text{ as } \ |x| \to +\infty\,,$$

where $c = (c_1, c_2, c_3)^\top$ is a constant vector.

Lemma 2. *The equation*

$$C(\partial)u(x) = \widetilde{P}(x), \ \ x \in \mathbb{R}^3\backslash\{0\},$$

has a unique homogeneous solution $u^{(0)} \in [C^\infty(\mathbb{R}^3\backslash\{0\})]^3$ of zero order satisfying the following condition

$$\int_{|x|=1} u^{(0)}(x)dS = 0.$$

The solution is representable in the form

$$u^{(0)}(x) := \mathcal{F}^{-1}_{\xi\to x}\big(v.p.\,[C(-i\xi)]^{-1}\mathcal{F}\,\widetilde{P}(\xi)\big), \tag{9}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are generalized Fourier transform operators and v.p. denotes that integrals are to be understood in the Cauchy principal sense.

Denote by $\Gamma_C = [\Gamma_{C,kj}]_{3\times 3}$ the fundamental solution of the operator $C(\partial)$ [2], [5].

Lemma 3. *Let $\widetilde{Q} = (Q_1, Q_2, Q_3)^\top \in [C^{0,\alpha}(\overline{\Omega^-})]^3 \cap [C^\infty(\Omega^-)]^3$ with*

$$\partial^\beta Q_j(x) = \mathcal{O}(|x|^{-3-|\beta|}) \ \text{ as } \ |x| \to \infty, \quad j = \overline{1,3},$$

for an arbitrary multi-index $\beta = (\beta_1, \beta_2, \beta_3)$, $|\beta| = \beta_1 + \beta_2 + \beta_3$.

Then the vector

$$v(x) = \int_{\Omega^-} \Gamma_C(x-y)\,\widetilde{Q}(y)\,dy,$$

is a particular solution of the equation

$$C(\partial)v(x) = \widetilde{Q}(x), \quad x \in \Omega^-.$$

Moreover, $v \in [C^\infty(\Omega^-)]^3 \cap [C^2(\overline{\Omega^-})]^3$ and

$$\partial^\beta v(x) = \mathcal{O}(|x|^{-1-|\beta|}\,\ln|x|\,) \ \text{ as } \ |x| \to \infty$$

for arbitrary multi-index $\beta = (\beta_1, \beta_2, \beta_3)$.

Corollary. Let $\widetilde{P}$ and $\widetilde{Q}$ be as above. Further, let $u \in [C^2(\Omega^-)]^3 \cap [C^1(\overline{\Omega^-})]^3$ be a solution of the equation

$$C(\partial)u(x) = \theta_0\,\widetilde{P}(x) + \widetilde{Q}(x) + \widetilde{\Phi}(x), \quad x \in \Omega^-$$

, satisfying the condition $u(x) = \mathcal{O}(1)$ as $|x| \to \infty$.

Then U can be represented as

$$u(x) = C + \theta_0\,u^{(0)}(x) + u^{(1)}(x),$$

where $c=(c_1,c_2,c_3)^\top$ is a constant vector, $u^{(0)}$ is given by (9) and

$$u^{(1)}\in[C^2(\Omega^-)]^3\cap[C^1(\overline{\Omega^-})]^3$$

possesses the following asymptotic at infinity

$$\partial^\beta u^{(1)}(x)=\mathcal{O}(|x|^{-1-|\beta|}\ln|x|\,)\ \text{ as }\ |x|\to\infty,$$

for an arbitrary multi-index $\beta=(\beta_1,\beta_2,\beta_3)$.
We can rewrite the equation (1) in the form

$$C(\partial)u(x)=\theta_0\widetilde{P}(x)+\widetilde{Q}(x)+\widetilde{\Phi}(x),\quad x\in\Omega^-.$$

Therefore its solution is representable as

$$u(x)=c+\theta_0u^{(0)}(x)+u^{(*)}(x),\quad x\in\Omega^-,$$

where $c=(c_1,c_2,c_3)^\top$ is a constant vector, $u^{(0)}$ is defined by (9), while

$$u^*\in[C^2(\Omega^-)]^3\cap[C^{1,\alpha}(\overline{\Omega^-})]^3\cap[C^\infty(\mathbb{R}^3\setminus\operatorname{supp}\Phi)]^3$$

and has the following asymptotic behaviour at infinity

$$\partial^\beta u^*(x)=\mathcal{O}(|x|^{-1-|\beta|}\ln(|x|)),\quad |x|\to\infty,$$

for arbitrary multi-index $\beta=(\beta_1,\beta_2,\beta_3)$.

Denote by $Z(\Omega^-)$ the class of vector functions from the space $[C^{1,\alpha}(\overline{\Omega^-})]^3$ satisfying the following relations:

$$\vartheta(x)=O(|x|^{-1}),\quad u(x)=O(1),\quad \lim_{R\to\infty}\frac{1}{4\pi R^2}\int_{\Sigma(0,R)}u(x)\,d\,\Sigma(0,R)=0,$$

where $\Sigma(0,R)$ is a sphere centered at the origin and radius R.

With the help of the above lemmata we can prove the following uniqueness result.

Theorem. *The exterior Dirichlet problem has at most one solution in the space* $[C^2(\Omega^-)]^3\cap[C^1(\overline{\Omega^-})]^3\cap Z(\Omega^-)$.

Similar uniqueness theorems hold true also for the exterior Neumann and mixed type boundary value problems for the static equations of the theory of thermoelasticity for anisotropic bodies.

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Authors' address:

M. Ivanidze and D. Ivanidze
Department of Mathematics
Georgian Technical University
77, M. Kostava St., Tbilisi 0175
Georgia
E-mail: marexiivanidze@mail.ru
diana-610@mail.ru