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BOUNDARY VALUE PROBLEMS OF THERMOSTATICS FOR HEMITROPIC ELASTIC SOLIDS

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Abstract. In the paper we consider the basic interior and exterior boundary value problems of the theory of thermoelasticty for hemitropic elastic solids. Applying the potential method and the theory of integral equations we prove the uniqueness and existence theorems.

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We investigate the basic boundary value problems of thermostatics for hemitropic elastic solids. Let $\Omega^+ \subset \mathbb{R}^3$ be a domain of finite diameter with the boundary $\partial\Omega^+ = S \in C^{2,\alpha}$, $0 < \alpha \le 1$; $\overline{\Omega^+} = \Omega^+ \cup S$ and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$.

The linear equations of statics of the hemitropic elasticity with regard to thermal effects read as [1], [2].

$$\begin{aligned}
&(\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,u(x) + (\chi+\nu)\Delta\omega(x) + \\
&\qquad (\delta+\chi-\nu)\,\mathrm{grad\,div}\,\omega(x) + 2\alpha\,\mathrm{curl}\,\omega(x) - \eta\,\mathrm{grad}\,\vartheta(x) = -\rho F(x),\\
&(\chi+\nu)\Delta u(x) + (\delta+\chi-\nu)\,\mathrm{grad\,div}\,u(x) + 2\alpha\,\mathrm{curl}\,u(x) + (\gamma+\varepsilon)\Delta\omega(x) + \\
&\qquad (\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\omega(x) + 4\nu\,\mathrm{curl}\,\omega(x) - \zeta\,\mathrm{grad}\,\vartheta(x) - 4\alpha\omega(x) = -\rho G(x),\\
&\kappa'\Delta\vartheta(x) = -Q(x),
\end{aligned} \tag{1}$$

where $u = (u_1, u_2, u_3)^\top$ is the displacement vector, $\omega = (\omega_1, \omega_2, \omega_3)^\top$ is the micro-rotation vector, ϑ is the temperature function, $-\rho F$ and $-\rho G$ are three dimensional given mass force and mass momentum vectors, and $-Q$ is a given heat source function; α, β, γ, δ, λ, μ, ν, $\varkappa$, ε, κ', $\eta > 0$ and $\zeta > 0$ are the material parameters.

Denote by $L(\partial)$ the matrix differential operator generated by equations (1),

$$L(\partial) = \begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) & L^{(5)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) & L^{(6)}(\partial) \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\Delta \end{bmatrix}_{7\times 7},$$

$$\begin{aligned}
&L^{(1)}(\partial) := (\mu+\alpha)\Delta I_3 + (\lambda+\mu-\alpha)Q(\partial),\\
&L^{(2)}(\partial) = L^{(3)}(\partial) := (\chi+\nu)\Delta I_3 + (\delta+\chi-\nu)Q(\partial) + 2\alpha R(\partial),\\
&L^{(4)}(\partial) := [(\gamma+\varepsilon)\Delta - 4\alpha]I_3 + (\beta+\alpha-\varepsilon)Q(\partial) + 4\nu\mathbb{R}(\partial),\\
&L^{(5)}(\partial) := -\eta\nabla^\top, \qquad L^{(6)}(\partial) := -\zeta\nabla^\top.
\end{aligned}$$

Introduce also the so called generalized stress operators $P(\partial, n)$ and $P^{*}(\partial, n)$ [2]:

$$\mathcal{P}(\partial, n)=\left[\begin{array}{ccc} T^{(1)}(\partial, n) & T^{(2)}(\partial, n) & -\eta\, n^{\top} \\ T^{(3)}(\partial, n) & T^{(4)}(\partial, n) & -\zeta\, n^{\top} \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\,\partial_n \end{array}\right]_{7\times 7},$$

$$\mathcal{P}^{*}(\partial, n)=\left[\begin{array}{ccc} T^{(1)}(\partial, n) & T^{(2)}(\partial, n) & -i\,\sigma\,\eta\, n^{\top} \\ T^{(3)}(\partial, n) & T^{(4)}(\partial, n) & -i\,\sigma\,\zeta\, n^{\top} \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\,\partial_n \end{array}\right]_{7\times 7},$$

where $n=(n_1, n_2, n_3)$ stands for the exterior unit normal vector to S,

$$\begin{aligned}
& T^{(j)}=\big[T_{pq}^{(j)}\big]_{3\times 3}, \quad j=\overline{1,4}, \\
& T_{pq}^{(1)}(\partial, n)=(\mu+\alpha)\delta_{pq}\partial_n+(\mu-\alpha)n_q\partial_p+\lambda n_p\partial_q, \\
& T_{pq}^{(2)}(\partial, n)=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k, \\
& T_{pq}^{(3)}(\partial, n)=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q, \\
& T_{pq}^{(4)}(\partial, n)=(\gamma+\varepsilon)\delta_{pq}\partial_n+(\gamma-\varepsilon)n_q\partial_p+\beta n_p\partial_q-2\nu\sum_{k=1}^{3}\varepsilon_{pqk}n_k.
\end{aligned}$$

Then equations (1) can be rewritten as

$$L(\partial)U(x)=\Phi(x), \quad x\in\Omega^{\pm}, \tag{2}$$

where we assume that

$$\begin{aligned}
& U=(u, \omega, \vartheta)^{\top}\in[C^{1,\beta}(\overline{\Omega^{\pm}})]^{7}\cap[C^{2}(\Omega^{\pm})]^{7}, \quad 0<\beta<\alpha\leq 1, \\
& \Phi=(\Phi_1, \Phi_2, ..., \Phi_7)^{\top}\in[C^{0,\beta}(\overline{\Omega^{\pm}})]^{7}.
\end{aligned}$$

Further we introduce the single and double layer and Newtonian potentials

$$\begin{aligned}
V(g)(x) &:= \int_S \Gamma(x-y)g(y)dS_y, \quad x\in\mathbb{R}^3\setminus S, \\
W(h)(x) &:= \int_S [P^{*}(\partial_y, n(y))\Gamma^{\top}(x-y)]^{\top}h(y)dS_y, \quad x\in\mathbb{R}^3\setminus S, \\
N_{\Omega^{\pm}}(\Psi)(x) &:= \int_{\Omega^{\pm}} \Gamma(x-y)\Psi(y)dy, \quad x\in\mathbb{R}^3,
\end{aligned}$$

where $\Gamma(x-y)=[\Gamma_{kj}(x-y)]_{7\times 7}$ is the matrix of fundamental solutions of the operator $L(\partial)$ constructed in [2], $g=(g_1, g_2, ..., g_7)^{\top}$ and $h=(h_1, h_2, ..., h_7)^{\top}$ are density vectors defined on S and $\Psi=(\Psi_1, \Psi_2, ..., \Psi_7)^{\top}$ is defined on $\Omega^{\pm}$ respectively. We assume that the vector-function Ψ has a compact support in the case of the domain Ω^{-}.

We say that a vector function $U=(u, \omega, \vartheta)^{\top}$ belongs to the class $Z(\Omega^{-})$ if

$$\begin{aligned}
& u(x)=\mathcal{O}(1), \quad \omega(x)=\mathcal{O}(|x|^{-2}), \quad \vartheta(x)=\mathcal{O}(|x|^{-1}), \\
& \lim_{R\to\infty}\tfrac{1}{4\pi R^2}\int_{\sum(0,R)} u(x)d\sum(0,R)=0, \quad x\in\Omega^{-},
\end{aligned}$$

where $\Sigma(0,R)$ is a sphere centered at the origin and radius R.

Lemma. *The double and single layer potentials, $V(g)$ and $W(h)$ solve the homogeneous equation $L(\partial)U(x)=0$ in $\mathbb{R}^3\setminus S$ and belong to the class $Z(\Omega^-)\cap C^{1,\beta}(\overline{\Omega^\pm})\cap C^\infty(\Omega^\pm)$ for $g\in[C^{0,\beta}(S)]^7$ and $h\in[C^{1,\beta}(S)]^7$. Moreover, there hold the following jump relations on S:*

$$\{V(g)(x)\}^\pm = V(g)(x)=\mathcal{H}g(x),\quad \{P(\partial_x,n(x))V(g)(x)\}^\pm=[\mp 2^{-1}I_7+\mathcal{K}]g(x),$$
$$\{W(h)(x)\}^\pm=[\pm 2^{-1}I_7+\mathcal{N}]h(x),$$
$$\{P(\partial_x,n(x))W(h)(x)\}^+ = P(\partial_x,n(x))W(h)(x)^- =: \mathcal{L}h(x),$$

where $x\in S$ and

$$\mathcal{H}g(x):=\int_S \Gamma(x-y)g(y)dS_y,\quad \mathcal{K}g(x):=\int_S [P(\partial_x,n(x))\Gamma(x-y)]g(y)dS_y,$$
$$\mathcal{N}h(x):=\int_S [P^*(\partial_y,n(y))\Gamma^\top(x-y)]^\top h(y)dS_y,$$
$$\mathcal{L}h(x):=\lim_{\Omega^\pm\ni z\to x\in S} P(\partial_z,n(z))\int_S [P^*(\partial_y,n(y))\Gamma^\top(x-y)]^\top h(y)dS_y.$$

Note that the Newtonian potential $U^{(0)}(x)=N_{\Omega^+}(\Phi)(x)$ is a particular solution of the nonhomogeneous differential equation (2) and belongs to the space $[C^{1,\beta}(\overline{\Omega^\pm})]^7\cap[C^2(\Omega^\pm)]^7\cap Z(\Omega^-)$ for $\Phi\in[C^{0,\beta}_{comp}(\overline{\Omega^\pm})]^7$. Therefore we can formulate the basic BVPs for the homogeneous differential equation. In particular, we can set the interior Dirichlet problem $(D)^+$ as follows: Find a solution $U=(u,\omega,\vartheta)^\top\in[C^{1,\beta}(\overline{\Omega^+})]^7\cap[C^2(\Omega^+)]^7$ to the equation $L(\partial)U(x)=0,\ \ x\in\Omega^+$, which satisfies the boundary condition on S $\{U(x)\}^+=f(x),\ \ x\in S$, where $f=(f_1,f_2,...,f_7)^\top\in[C^{1,\beta}(S)]^7$.

There hold the following existence results for the problem $(D)^+$.

Theorem 1. *Let $S\in C^{2,\alpha}$ and $f\in[C^{1,\beta}(S)]^7$, $0<\beta<\alpha\le 1$. Then the Dirichlet problem $(D)^+$ is uniquely solvable in the space $[C^{1,\beta}(\overline{\Omega^+})]^7\cap[C^2(\overline{\Omega^+})]^7$ and the solution is representable in the form of a double layer potential $U=W(g)$ where the density vector $g\in[C^{1,\beta}(S)]^7$ is uniquely defined by the singular integral equation*

$$2^{-1}g(x)+\mathcal{N}g(x)=f(x),\quad x\in S.$$

Theorem 2. *Let $S\in C^{2,\alpha}$ and $f\in[C^{1,\beta}(S)]^7, 0<\beta<\alpha\le 1$. Then the Dirichlet problem $(D)^+$ is uniquely solvable in the space $[C^{1,\beta}(\overline{\Omega^+})]^7\cap[C^2(\overline{\Omega^+})]^7$ and the solution is representable in the form of a single layer potential $U=V(g)$ where the density vector $g\in[C^{1,\beta}(S)]^7$ is uniquely defined by the integral equation*

$$\mathcal{H}g(x)=f(x),\quad x\in S.$$

Now we formulate the exterior Dirichlet problem $(D)^-$:
Find a solution $U\in[C^{1,\beta}(\overline{\Omega^-})]^7\cap[C^2(\Omega^-)]^7\cap Z(\Omega^-)$, $0<\beta<\alpha\le 1$, to the equation

$$L(\partial)U(x)=0,\quad x\in\Omega^-, \tag{3}$$

satisfying the boundary condition

$$\{U(x)\}^-=f(x),\quad x\in S,$$

where $f = (f_1, f_2, ..., f_7)^\top \in [C^{1,\beta}(S)]^7$.

We have the following existence results for the problem $(D)^-$.

Theorem 3. *Let $S \in C^{2,\alpha}$ and $f \in [C^{1,\beta}(S)]^7$, $0 < \beta < \alpha \leq 1$. Then the Dirichlet problem $(D)^-$ is uniquely solvable in the space $[C^{1,\beta}(\overline{\Omega^-})]^7 \cap [C^2(\overline{\Omega^-})]^7 \cap Z(\Omega^-)$ and the solution is representable in the form of a linear combination of double and single layer potentials $U = W(g) + a\,V(g)$ with a positive constant $a > 0$, where the density vector $g \in [C^{1,\beta}(S)]^7$ is uniquely defined by the singular integral equation*

$$-2^{-1}\,g(x) + \mathcal{N}g(x) + a\,\mathcal{H}g(x) = f(x), \quad x \in S.$$

Theorem 4. *Let $S \in C^{2,\alpha}$ and $f \in [C^{1,\beta}(S)]^7, 0 < \beta < \alpha \leq 1$. Then the Dirichlet problem $(D)^-$ is uniquely solvable in the space $[C^{1,\beta}(\overline{\Omega^-})]^7 \cap [C^2(\overline{\Omega^-})]^7 \cap Z(\Omega^-)$ and the solution is representable in the form of a single layer potential $U = V(g)$ where the density vector $g \in [C^{1,\beta}(S)]^7$ is uniquely defined by the integral equation*

$$\mathcal{H}g(x) = f(x), \quad x \in S.$$

Further we formulate the exterior Neumann problem $(N)^-$:
Find a solution $U \in [C^{1,\beta}(\overline{\Omega^-})]^7 \cap [C^2(\Omega^-)]^7 \cap Z(\Omega^-)$, $0 < \beta < \alpha \leq 1$, to equation (3) satisfying the boundary condition

$$\{P(\partial, n)U(x)\}^- = F(x), \qquad x \in S,$$

where $F = (F_1, F_2, ..., F_7)^\top \in [C^{0,\beta}(S)]^7$.

In this case we have the following existence result.

Theorem 5. *Let $S \in C^{1,\alpha}$ and $F \in [C^{0,\beta}(S)]^7$, $0 < \beta < \alpha \leq 1$. Then the Neumann problem $(N)^-$ is uniquely solvable in the space $[C^{1,\beta}(\overline{\Omega^-})]^7 \cap [C^2(\overline{\Omega^-})]^7 \cap Z(\Omega^-)$ and the solution is representable in the form of a single layer potentials $U = V(g)$, where the density vector $g \in [C^{0,\beta}(S)]^7$ is uniquely defined by the singular integral equation*

$$2^{-1}\,g(x) + \mathcal{K}g(x) = F(x), \quad x \in S.$$

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