Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 24, 2010

NUMERICAL RESOLUTION OF ONE NONLINEAR PARABOLIC SYSTEM

Gagoshidze M.

Abstract. Numerical resolution of one nonlinear system of parabolic equations is studied. Considered model is the one-dimensional analog of Maxwell's system which describes process of penetration of magnetic field into a substance. Graphs of numerical experiments based on constructed finite difference schemes are given.

Keywords and phrases: Nonlinear parabolic system, finite difference schemes.

AMS subject classification: 35Q60, 35K55, 65M06.

Let us consider following initial-boundary value problem for one-dimensional analog of the system of Maxwell's equations which describes process of penetration of the magnetic field into a substance [1]:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(a(V)\frac{\partial U}{\partial x}\right), \qquad (x,t) \in \Omega \times (0,T), \tag{1}$$

$$\frac{\partial V}{\partial t} = a(V)\left(\frac{\partial U}{\partial x}\right)^2 + \varepsilon\frac{\partial^2 V}{\partial x^2}, \quad (x,t) \in \Omega \times (0,T), \tag{2}$$

$$U(x,t) = V(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{3}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x) \geq Const > 0, \qquad x \in \overline{\Omega}, \tag{4}$$

where a, U_0, V_0 are known functions of their arguments; ε, T are the fixed positive constants and $\overline{\Omega} = (0,1)$.

By the first term in the right hand side of the equation (2) Joule's rule is considered and by the second term in the same equation thermal conductivity is described.

The problem with taking into account only Joule's rule ($\varepsilon = 0$) as well as system with both physical terms ($\varepsilon > 0$) are considered by many authors (see, for example, [2]-[8] and references therein).

Note that, system (1), (2) without thermal conductivity can be reduced to integro-differential form [9]. Many works are devoted to the integro-differential models of these type (see, for example, [10]-[20] and references therein). The questions of existence, uniqueness, asymptotic behavior of the solutions and numerical resolution of some kind of initial-boundary value problems for this type integro-differential models are studied in these works.

The purpose of the present note is construction of the finite difference scheme for (1)-(4) problem.

Let us introduce the uniform grids $\overline{\omega}_h = \{t_i = ih,\ i = 0, 1, ..., M\}$ on $[0,1]$ and $\omega_\tau = \{t_j = j\tau,\ \ j = 0, 1, ..., N\}$ on $[0,T]$.

Using usual notations and technique of building of the finite difference schemes (see, for example, [21]) let us construct following approximate models for the problem (1)-(4).

1. Semi-implicit finite difference scheme. At first let us correspond to problem (1)-(4) the following so called semi-implicit scheme:

$$\frac{u_i^{j+1}-u_i^j}{\tau}=\frac{1}{h^2}\left\{a\left(v_i^{j+1}\right)u_{i-1}^{j+1}\right.$$

$$\left.-\left(a\left(v_i^{j+1}\right)+a\left(u_{i+1}^{j+1}\right)\right)u_i^{j+1}+a\left(v_{i+1}^{j+1}\right)u_{i+1}^{j+1}\right\}+f_i^{j+1}, \tag{5}$$

$$\frac{v_i^{j+1}-v_i^j}{\tau}=a\left(v_i^j\right)\left(\frac{u_i^j-u_{i-1}^j}{h}\right)^2+\varepsilon\frac{v_{i+1}^{j+1}-2v_i^{j+1}+v_{i-1}^{j+1}}{h^2}+g_i^{j+1}, \tag{6}$$

$$i=1,2...M-1;\quad j=0,1...,N-1,$$

$$u_0^j=u_M^j=v_0^j=v_M^j=0,\quad j=0,1,...,N, \tag{7}$$

$$u_j^0=U_{0,i}\quad v_j^0=V_{0,i},\quad i=0,1...,M. \tag{8}$$

Using the tridiagonal matrix algorithm in the first step from (6)-(8) we find the second component v of the approximate solution of the scheme (5)-(8). In the second step by using once again the tridiagonal matrix algorithm from (5),(7),(8) we will find u.

2. Implicit finite difference scheme. In this case in the scheme (5)-(8) instead of equation (6) we have

$$\frac{v_i^{j+1}-v_i^j}{\tau}=a\left(v_i^j\right)\left(\frac{u_i^{j+1}-u_{i-1}^{j+1}}{h}\right)^2+\varepsilon\frac{v_{i+1}^{j+1}-2v_i^{j+1}+v_{i-1}^{j+1}}{h^2}+g_i^{j+1}. \tag{9}$$

For the solving of (5), (7), (9) model Newton iterative algorithm is used [22].

Many numerical test experiments are carried out on the basis of these constructed discrete analogues.

The numerical experiments are quite satisfactory and fully agree with the considered exact test solutions of problem (1)-(4). One of these solutions have the form:

$$U(x,t)=\frac{1}{2}\sin^2(\pi x)(1+t),\quad V(x,t)=\frac{1}{4}\sin^2(\pi x)(1+t^2).$$

The graphs of suitable numerical results are given on the Figs. 1-3.

Let us note that numerical experiments give convergence of the considered schemes when $\tau\to 0$, $h\to 0$. The convergence effect when $\varepsilon\to 0$ is also established by numerical experiments. Particularly, the numerical experiments show us that the solutions of problem (1)-(4) tend to the solution of same problem with $\varepsilon=0$ when $\varepsilon\to 0$.

Test 1. Let us consider following nonlinearity $a(V)=(1+V)^{1/2}$. In this test the numerical computations are carried out by using semi-implicit scheme (5)-(8): Fig. 1.

Test 2. In this test we consider following nonlinearity $a(V)=\frac{1}{1+V^{1/2}}$, and numerical computations are carried out again by using semi-implicit scheme (5)-(8): Fig. 2.

Test 3. In this test we consider the same nonlinearity as in test 2, $a(V)=\frac{1}{1+V^{1/2}}$, and numerical computations are carried out by implicit scheme using iterative method: Fig. 3.

To solve system of (5), (7), (8), (9) equations we are computing solution of (7)-(9) system and then we are computing solution of (5), (7), (8). We move up to new layer when the difference between 2 consequent iteration is less then 0.0001.

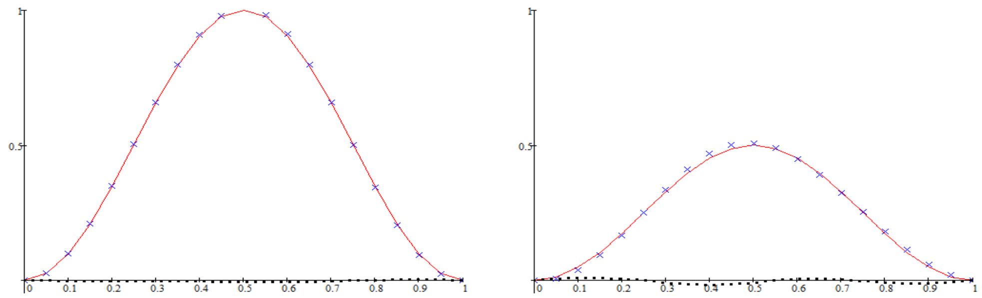

Fig. 1. Exact (solid line) and numerical (marked with $\times$) solutions and differences between exact and numerical solutions (marked with $\bullet$).

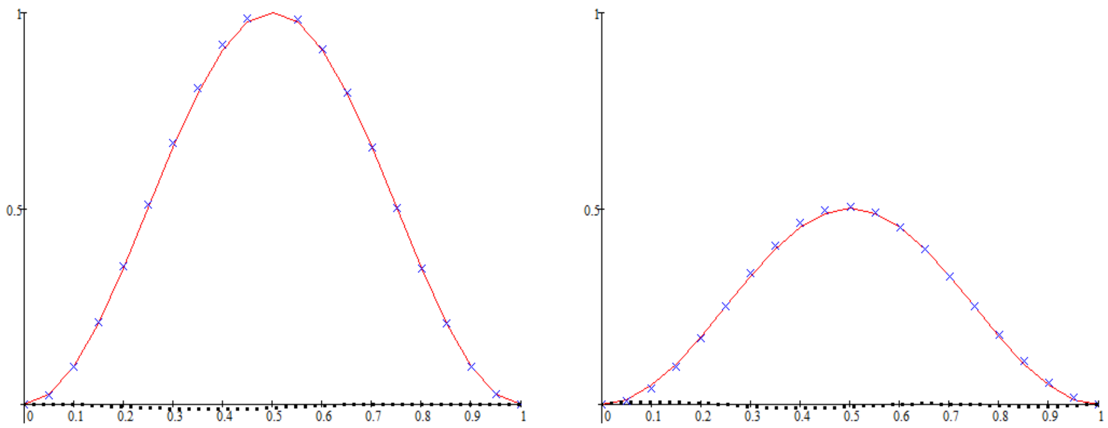

Fig. 2. Exact (solid line) and numerical (marked with $\times$) solutions and differences between exact and numerical solutions (marked with $\bullet$).

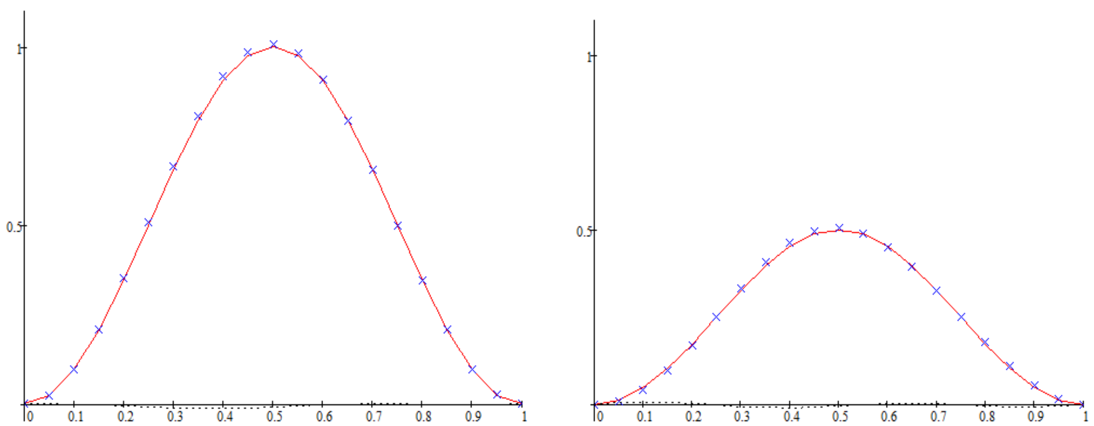

Fig. 3. Exact (solid line) and numerical (marked with $\times$) solutions and differences between exact and numerical solutions (marked with $\bullet$).

R E F E R E N C E S

1. Landau L., Lifschitz E. Electrodynamics of Continuous Media. (Russian) *Moscow*, 1958.

2. Dafermos C.M., Hsiao L., Adiabatic shearing of incompressible fluids with temperaturedependent viscosity. *Quart. Appl. Math.,* **41** (1983), 45-58.

3. Abuladze I.O., Gordeziani D.G., Dzhangveladze T.A., Korshiya T.K. Discrete models of a nonlinear magnetic-field-scattering problem with thermal conductivity. (Russian) *Differ. Uravn.*, **22**, 7 (1986), 1119-1129. English translation: *Differ. Equ.*, **22**, 7 (1986), 769-777.

4. Dzhangveladze T. The difference scheme for one system of nonlinear partial differential equations. (Russian) *Rep. Enlarged Sess. Semin. I.Vekua Appl. Math.*, **2**, 3 (1986), 40-43.

5. Yin H.-M. Global solutions of Maxwell's equations in an electromagnetic field with a temperature-dependent electrical conductivity. *European J. Appl. Math.,* **5**, 1 (1994), 57-64.

6. Bien M. Existence of global weak solutions for a class of quasilinear equations describing Joule's heating. Math. Meth. Appl. Sci., **23** (1998), 1275–1291.

7. Kiguradze Z. The difference scheme for one system of nonlinear partial differential equations. *Rep. Enlarged Sess. Semin. I.Vekua Appl. Math.*, **14**, 3 (1999), 67-70.

8. Sun D., Manoranjan V.S., Yin H.-M. Numerical solutions for a coupled parabolic equations arising induction heating processes. *Discrete Contin. Dyn. Syst., Supplement,* (2007), 956-964.

9. Gordeziani D.G., Dzhangveladze T.A., Korshia T.K. Existence and uniqueness of the solution of a class of nonlinear parabolic problems. *Differ. Uravn.*, **19**, 7 (1983), 1197-1207 (Russian). English translation: *Differ. Equ.*, **19** (1984), 887-895.

10. Dzhangveladze T.A. First boundary-value problem for a nonlinear equation of parabolic type. (Russian) *Dokl. Akad. Nauk SSSR*, **269**, 4 (1983), 839-842. English translation: *Soviet Phys. Dokl.*, **28**, 4 (1983), 323-324.

11. Laptev G.I. Mathematical singularities of a problem on the penetration of a magnetic field into a substance. (Russian) *Zh. Vychisl. Mat. Mat. Fiz.,* **28** (1988), 1332–1345. English translation: *U.S.S.R. Comput. Math. Math. Phys.* **28** (1990), 35–45.

12. Lin Y., Yin H.-M. Nonlinear parabolic equations with nonlinear functionals. *J. Math. Anal. Appl.*, **168**, 1 (1992), 28-41.

13. Jangveladze T. Convergence of a difference scheme for a nonlinear integro-differential equation. Proc. of I.Vekua Inst. Appl. Math., **48** (1998), 38-43.

14. Jangveladze T., Kiguradze Z. Estimates of a stabilization rate as $t \to \infty$ of solutions of a nonlinear integro-differential equation. *Georgian Math. J.*, **9**, 1 (2002), 57-70.

15. Jangveladze T., Kiguradze Z. Estimates of a stabilization rate as $t \to \infty$ of solutions of a nonlinear integro-differential diffusion system. *J. Appl. Math. Inform. Mech.*, **8**, 2 (2003), 1-19.

16. Jangveladze T., Kiguradze Z. Asymptotics of a solution of a nonlinear system of diffusion of a magnetic field into a substance. (Russian) *Sibirsk. Mat. Zh.*, **47**, 5 (2006), 1058-1070. English translation: *Siberian Math. J.*, **47**, 5 (2006), 867-878.

17. Dzhangveladze T.A., Kiguradze Z.V. Asymptotic behavior of the solution to nonlinear integro-differential diffusion equation. (Russian) *Differ. Uravn.*, **44**, 4 (2008), 517-529. English translation:*Differ. Equ.*, **44**, 4 (2008), 538-550.

18. Aptsiauri M., Jangveladze T., Kiguradze Z. Large time behavior of solutions and numerical approximation of nonlinear integro-differential equation associated with the penetration of a magnetic field into a substance. *J. Appl. Math. Inform. Mech.*, **13**, 2 (2008), 3-17.

19. Jangveladze T., Kiguradze Z., Neta B. Large time behavior of solutions to a nonlinear integro-differential system. *J. Math. Anal. Appl.*, **351**, 1 (2009), 382-391.

20. Jangveladze T., Kiguradze Z., Neta B. Finite difference approximation of a nonlinear integro-differential system. *Appl. Math.Comput.*, **215**, 2 (2009), 615-628.

21. Samarskii A.A. The Theory of Difference Schemes. (Russian) *Moscow,* 1977.

22. Rheinboldt W.C. Methods for Solving Systems of Nonlinear Equations. *SIAM, Philadelphia*,

1970.

Received 12.05.2010; revised 12.09.2010; accepted 15.10.2010.

Author's address:

M. Gagoshidze
Ilia State University
Faculty of Physics and Mathematics
32, Chavchavadze Av., Tbilisi 0179
Georgia
E-mail: MishaGagoshidze@gmail.com