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NUMERICAL CALCULATIONS OF THE KIRCHHOFF NONLINEAR DYNAMIC BEAM

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Abstract. An initial value problem is posed for the Kirchhoff integro-differential equation, which describes the dynamic state of a beam. The solution is approximated with respect to a spatial and a time variables by the Galerkin method and a difference scheme. The algorithm has been approved on tests and the results of recounts are represented in graphics.

Keywords and phrases: Kirchhoff nonlinear dynamic beam equation, approximate algorithm, Galerkin's method, difference scheme, calculations results.

AMS subject classification: 65M60, 65N15.

1. Statement of the problem. Let us consider the nonlinear integro-differential equation

$$\frac{\partial^2 u}{\partial t^2}(x,t)+\frac{\partial^4 u}{\partial x^4}(x,t)-\left(\alpha+\beta\int\limits_0^L\left(\frac{\partial u}{\partial x}(x,t)\right)^2dx\right)\frac{\partial^2 u}{\partial x^2}(x,t)=f(x,t), \tag{1}$$

$$0<x<L,\ \ 0<t\le T,$$

with the initial boundary conditions:

$$u(x,0)=u^0(x),\qquad \frac{\partial u}{\partial t}(x,0)=u^1(x), \tag{2}$$

$$u(0,t)=u(L,t)=0,\qquad \frac{\partial^2 u}{\partial x^2}(0,t)=\frac{\partial^2 u}{\partial x^2}(L,t)=0, \tag{3}$$
$$0\le x\le L,\ \ 0\le t\le T,$$

where α,β,L and T are some positive constants, $f(x,t),u^0(x)$ and $u^1(x)$ are the given functions, and $u(x,t)$ is the function we want to define. Equation (1) describes the oscillation of a beam in the light of Kirchhoff theory. Several authors dedicated their works to the study of this equation (see, for example [1-11]).

2. The algorithm. The algorithm consists of two parts.

First part-the Galerkin method.

The solution of problem (1)-(3) is represented in the form

$$u_n(x,t)=\sum_{i=1}^n u_{ni}(t)\sin\frac{i\pi}{L}x. \tag{4}$$

where the coefficients $u_{ni}(x)$ satisfy the following system of differential equations

$$u''_{ni}(t)+\left(\frac{i\pi}{L}\right)^4 u_{ni}(t)+\left(\frac{i\pi}{L}\right)^2\left(\alpha+\beta\frac{\pi}{2L}\sum_{j=1}^{n} j^2 u_{nj}^2(t)\right)u_{ni}(t)=f_i(t), \qquad (5)$$

$$i=1,2,\cdots,n,\quad 0<t\le T,$$

with the initial conditions

$$u_{ni}(0)=a_i^{(0)},\ \ u'_{ni}(0)=a_i^{(1)},\quad i=1,2,\cdots,n, \qquad (6)$$

where

$$a_i^{(p)}=\frac{2}{L}\int_0^L u^{(p)}(x)\sin\frac{i\pi}{L}x dx,\quad p=0,1,$$

$$f_i(t)=\frac{2}{L}\int_0^L f(x,t)\sin\frac{i\pi}{L}x dx,\quad i=1,2,\cdots,n.$$

Second part-a difference scheme (see [12]).

We proceed to solving problem (5), (6) by means of the difference method. On the time interval $[0,T]$ let us introduce the grid $\{t_k|0=t_0<t_1<\cdots<t_m=T\}$ with a step $\tau=T/m$, $t_k=k\tau$, $k=0,1,\cdots,m$.

Use the explicit symmetric difference scheme:

$$\frac{u_{ni}^{k+1}-2u_{ni}^k+u_{ni}^{k-1}}{\tau^2}$$
$$+\left(\frac{i\pi}{L}\right)^2\left(\left(\frac{i\pi}{L}\right)^2+\alpha+\beta\frac{\pi^2}{2L}\sum_{j=1}^{n} j^2\left(u_{nj}^k\right)^2\right)\frac{u_{ni}^{k+1}+u_{ni}^{k-1}}{2}=f_i^{(k)},$$
$$k=1,2,\cdots,m-1,$$

where $u_{ni}^k=u_{ni}(t_k)$, $f_i^{(k)}=f_i(t_k)$, $k=0,1,\cdots,m$. On the first two levels let us use formulas

$$u_{ni}^0=a_i^{(0)},$$
$$u_{ni}^1=a_i^{(0)}+\tau a_i^{(1)}-\frac{\tau^2}{2}\left(\left(\frac{i\pi}{L}\right)^2+\alpha+\beta\frac{\pi^2}{2L}\sum_{j=1}^{n} j^2\left(a_j^{(0)}\right)^2\right)\left(\frac{i\pi}{L}\right)^2 a_i^{(0)},$$
$$i=1,2,\cdots,n.$$

The other layers to use the following formula:

$$u_{ni}^{k+1}=\frac{2u_{ni}^k+\tau^2 f_i^{(k)}}{1+\frac{\tau^2}{2}\left(\frac{i\pi}{L}\right)^2\left[\left(\frac{i\pi}{L}\right)^2+\alpha+\beta\frac{\pi^2}{2L}\sum_{j=1}^{n} j^2\left(u_{nj}^k\right)^2\right]}-u_{ni}^{k-1},$$

$$k=1,2,\cdots,m-1.$$

3. Algorithms realization. The algorithm proposed in subsection 2 enables us to find approximate solutions of problems (1)-(3). The approximate program has been designed in Turbo Pascal algorithm language and calculations have been done on the computer. The results obtained are good enough. The algorithm has been approved tests and the results of recounts are represented in graphics. The algorithm is approved in the following two tasks on the test:

a). test - 1.

an exact solution $u(x,t) = \sin\frac{\pi x}{L} + t\sin\frac{2\pi x}{L}$,

the given functions $u^{(0)}(x) = \sin\frac{\pi x}{L}$, $u^{(1)}(x) = \sin\frac{2\pi x}{L}$,

the right hand side

$$f(x,t) = \frac{\pi^4}{L^4}\sin\frac{\pi x}{L} + \frac{16\pi^4}{L^4}t\sin\frac{2\pi x}{L} + \left(\alpha + \beta\frac{\pi^2}{2L}(1+4t^2)\right)\frac{\pi^2}{L^2}\left(\sin\frac{\pi x}{L} + 4t\sin\frac{2\pi x}{L}\right),$$

b). test - 2.

an exact solution $u(x,t) = \sin\frac{\pi x}{L} + t^3\sin\frac{2\pi x}{L}$,

the given functions $u^{(0)}(x) = \sin\frac{\pi x}{L}$, $u^{(1)}(x) = 0$,

the right hand side

$$f(x,t) = 6t\sin\frac{2\pi x}{L} + \frac{\pi^4}{L^4}\sin\frac{\pi x}{L} + \frac{16\pi^4}{L^4}t^3\sin\frac{2\pi x}{L} +$$

$$+\left(\alpha + \beta\frac{\pi^2}{2L}(1+4t^6)\right)\frac{\pi^2}{L^2}\left(\sin\frac{\pi x}{L} + 4t^3\sin\frac{2\pi x}{L}\right),$$

numerical parameters $\alpha = 1,\ \beta = 1,\ n = 5,\ m = 10,\ l = 10.$

The exact solutions are drawing on the Fig. 1 and Fig. 3. Approximate solutions were performed according to formula (4) and are given on the Fig. 2 and Fig. 4.

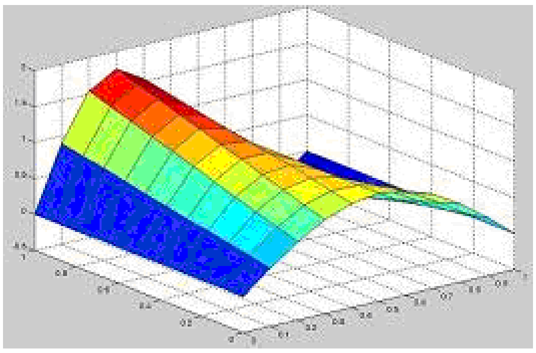

Fig.1.

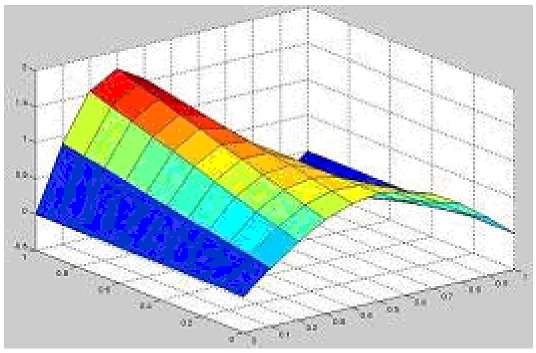

Fig.2

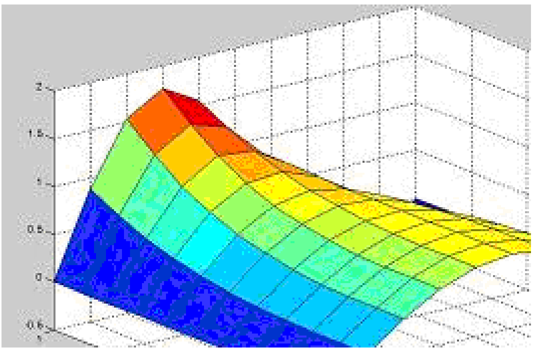

Fig.3.

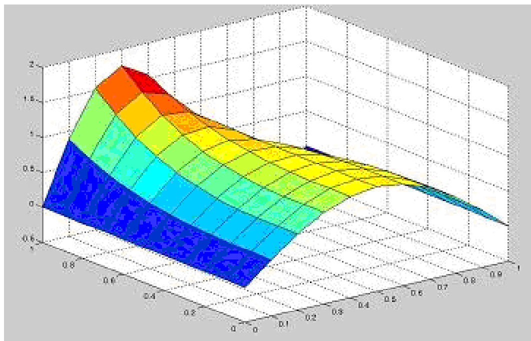

Fig.4

Naturally, exact and approached decisions coincide with each other in case of the test1 (see Fig. 1, 2), and in case of the test 2 (see Fig. 3, 4) they differ from each other slightly with accuracy $0(\tau^2)$.

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