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ON ONE PROBLEM OF HYPOTHESES TESTING

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Abstract. The problem of testing two simple hypotheses for a Gaussian Markovian process is reduced to an optimal stopping problem for a two-dimensional random Markovian process. The latter problem is reduced in turn to the corresponding Stefan problem. A solution of a second order differential equation is found for the Stefan problem.

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1. Assume that we observe a random process $\xi = (\xi_t)$, $t \geq 0$, that satisfies the stochastic differential equation

$$d\xi_t = -r\theta\xi_t\, dt + \sigma\, dw_t, \quad \sigma > 0, \quad r \neq 0,$$

where $w = (w_t)$ is a standard Wiener process and θ is an unknown random variable. Also assume that we have a given family of probability measures $\{P_\pi,\ 0 \leq \pi \leq 1\}$ such that

$$P_\pi = \pi P_1 + (1-\pi)P_0.$$

Let a random value θ takes two values 1 and 0 with probabilities $P_\pi(\theta = 1) = \pi$ and $P_\pi(\theta = 0) = 1-\pi$. The problem consists in determining the true value of the parameter θ by observation of the process ξ, i.e., in checking which of the two hypotheses is true: $H_0:\ \theta = 0$ or $H_1:\ \theta = 1$.

Let $\delta = (\tau, d)$ be a decision function (decision rule), where τ is a moment of time at which the observation was stopped (a stopping time) and d is the decision made at the moment τ and taking two values 1 or 0 [1]. If $d = 1$, then the hypothesis H_1 is accepted, but if $d = 0$, then the hypothesis H_0.

2. Let α and β denote the first and second order error probabilities

$$\alpha = P_1(d = 0), \quad \beta = P_0(d = 1).$$

Assume that an average loss for the decision rule $\delta = (\tau, d)$ is measured by the value

$$\rho_\delta(\pi) = \pi\big[cE_1\tau + aP_1(d=0)\big] + (1-\pi)\big[cE_0\tau + bP_0(d=1)\big], \tag{1}$$

where a, b, c are non-negative constants, E_0 and E_1 are the values averaged with respect to the measures P_0 and P_1, respectively.

The decision rule $\delta^* = (\tau^*, d^*)$ is called π-Bayes if

$$\rho_{\delta^*}(\pi) = \inf_\delta \rho_\delta(\pi),$$

where the infimum is taken with respect to the class of all decision rules. The decision rule $\delta^* = (\tau^*, d^*)$ is called Bayes if δ^* is the π-Bayes rule for all $0 \leq \pi \leq 1$.

Denote by

$$\pi = P_\pi\big(\theta = 1|\ \mathcal{F}_t^\xi\big), \quad \mathcal{F}_t^\xi = \sigma\{\xi_s,\ s \le t\},$$

an a posteriori probability of the hypothesis $H_1:\ \theta = 1$ and assume that $\overline{w} = (\overline{w}_t)$ is the so-called innovation Wiener process [2], [3].

Lemma 1. *Random processes $\pi = (\pi_t)$ and $\xi = (\xi_t)$, $t \ge 0$, satisfy the following stochastic differential equations*

$$d\pi_t = \frac{r}{\sigma}\,\pi_t(1-\pi_t)\xi_t\,d\overline{w}_t, \tag{2}$$

$$d\xi_t = -r\pi_t\xi_t\,dt + \sigma\,d\overline{w}_t. \tag{3}$$

Lemma 2. *A pair of processes $(\pi, \xi) = (\pi_t, \xi_t)$, $t \ge 0$, determined by the equations (2), (3) is a Markovian process.*

Lemma 3. *For a decision rule $\delta = (\tau, d)$ the value (1) is given as follows:*

$$\rho_\delta(\pi) = \rho_\delta(\pi, \xi) = g(\pi, \xi),$$

where

$$g(\pi, \xi) = c\tau + \min\big[a\pi + b(1-\pi)\big] + \lambda\xi(1-\pi), \quad \lambda > 0.$$

Lemma 4. *For any decision rule $\delta = (\tau, d)$ there exists a decision rule $\widetilde{\delta} = (\tau, \widetilde{d})$ such that*

$$\rho_{\widetilde{\delta}}(\pi, \xi) \le \rho_\delta(\pi, \xi)$$

while the value $\rho(\pi, \xi)$ of observation of a process $(\pi, \xi) = (\pi_t, \xi_t)$, $t \ge 0$, is a solution of the following optimal stopping problem

$$\rho(\pi, \xi) = \inf_\delta \rho_\delta(\pi, \xi) = \inf_\tau E_{\pi,\xi} g(\pi_\tau, \xi_\tau), \tag{4}$$

where τ is a stopping time from the class $\mathfrak{M}^{\pi,\xi}$ with respect to the σ-algebra $\mathcal{F}^{\pi,\xi} = \sigma\{(\pi_s, \xi_s),\ s \le t\}$.

By the foregoing lemmas it can be proved that the value $\rho(\pi, \xi)$ defined by means of (4) is a solution of the following Stefan problem.

Theorem 1. *On the set $D = \big\{(\pi, \xi):\ \rho(\pi, \xi) < g(\pi, \xi)\big\}$ the value $\rho(\pi, \xi)$ satisfies the differential equation*

$$\frac{1}{2}\frac{r^2}{\sigma^2}\,\pi^2(1-\pi)^2\xi^2\rho''_{\pi\pi} + \frac{1}{2}\,\sigma^2\rho''_{\xi\xi} - r\pi(1-\pi)\xi\rho''_{\pi\xi} - r\pi\xi\rho'_\xi = -c,$$

with the following boundary conditions on the boundary ∂D of the set D:

$$\rho\big|_{\partial D} = g\big|_{\partial D}, \quad \frac{\partial \rho}{\partial \pi}\Big|_{\partial D} = \frac{\partial g}{\partial \pi}\Big|_{\partial D}, \quad \frac{\partial \rho}{\partial \xi}\Big|_{\partial D} = \frac{\partial g}{\partial \xi}\Big|_{\partial D}.$$

We will seek for $\rho(\pi, \xi)$ in the form

$$\rho(\pi, \xi) = \sum_{k=0}^{\infty} f_k(\xi) \cdot \pi^k.$$

Now let us introduce the transformation

$$f_k(\xi) = u_k(x) \exp \frac{rk}{2\sigma^2} (1 - \sqrt{k}) x^2,$$

where

$$x = \sqrt{\frac{2rk}{\sigma^2}} (\xi + \sqrt{k} - 1).$$

It is not difficult to verify that for $k \geq 2$ the function $f_k(\xi)$ is a solution of the differential equation

$$\frac{1}{2} \sigma^2 f_k''(\xi) - rk\xi f_k'(\xi) + \frac{1}{2} \frac{r^2}{\sigma^2} k(k-1) f_k(\xi) = 0$$

while the function $u_k(x)$ is a solution of the Weber differential equation

$$u_k''(x) - x u_k'(x) - \frac{\sqrt{k} - 1}{2\sqrt{k}} u_k(x) = 0.$$

Theorem 2. *A function $f_k(\xi)$ is given by the expression*

$$f_k(\xi) = \exp \frac{rk}{2\sigma^2} (1 - \sqrt{k}) \xi^2$$

$$\times \left[1 + \sum_{i=1}^{\infty} \frac{s(s+2)\cdots(s+2i-2)}{(2i)!} \left(\frac{2rk}{\sigma^2} \right)^i \left(\xi + \sqrt{k} - 1 \right)^{2i} \right],$$

where

$$s = \frac{\sqrt{k} - 1}{2\sqrt{k}}.$$

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