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THE BASIC CONTACT PROBLEM FOR PIECE-WISE HOMOGENEOUS TRANSVERSALLY ISOTROPIC THERMOELASTIC PLANE

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Abstract. In the present paper an explicit solution of basic contact problem of thermoelasticity is constructed for the two-dimensional equations of thermoelastic transversally isotropic plane. For solution we use the potential method and constructed the special fundamental matrices, which reduced the contact problem to a Fredholm integral equations of the second kind. For the equation of statics of thermoelasticity we construct one particular solution and we reduce the solution of basic contact problem of the theory of thermoelasticity to the solution the basic contact problem for the equation of transversally-isotropic body. Poisson type formula for the solution of the basic contact problem for the plane is constructed.

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Let the plane is divided into two half-plane $x_3 < 0$ and $x_3 > 0$. Denote by $D^{(1)}$ the half-plane $x_3 > 0$, by $D^{(0)}$ the half-plane $x_3 < 0$ and by S (the axis ox_1) the interface $x_3 = 0$. Let the domains $D^{(j)}$ are filled by homogeneous transversally-isotropic materials with the coefficients $c_{pq}^{(j)}$, $j = 0, 1$.

We say that a body is subject to a plane deformation if the second component of the displacement vector equals to zero and the other components u_1, u_3 depend only on x_1, x_3. In this case the basic two-dimensional equations thermoelasticity for the transversally-isotropic body in the domains $D^{(j)}$ can be written as follows [1]

$$C^{(j)}(\partial x)U^{(j)} = B^{(j)} grad u_4^{(j)}, \tag{1}$$

$$\Delta_4^{(j)} u_4 = a_4^{(j)} \frac{\partial^2 u_4^{(j)}}{\partial x_1^2} + \frac{\partial^2 u_4^{(j)}}{\partial x_3^2} = 0, \quad j = 0, 1, \tag{2}$$

where

$$C^{(j)}(\partial x) = \|C_{pq}^{(j)}(\partial x)\|_{2x2}, \quad B^{(j)} = \|B_{pq}^{(j)}\|_{2x2}, \quad B_{11}^{(j)} = \beta_1^{(j)}, \quad B_{22}^{(j)} = \beta_2^{(j)},$$
$$B_{12}^{(j)} = B_{21}^{(j)} = 0, \quad C_{11}^{(j)}(\partial x) = c_{11} \frac{\partial^2}{\partial x_1^2} + c_{44}^{(j)} \frac{\partial^2}{\partial x_3^2}, \quad C_{21}^{(j)}(\partial x) =$$
$$C_{12}^{(j)}(\partial x) = (c_{13}^{(j)} + c_{44}^{(j)}) \frac{\partial^2}{\partial x_1 \partial x_3}, \quad C_{22}^{(j)}(\partial x) = c_{11}^{(j)} \frac{\partial^2}{\partial x_1^2} + c_{44}^{(j)} \frac{\partial^2}{\partial x_3^2},$$

$c_{pq}^{(j)}$ are Hooke's coefficients, $\beta_1^{(j)} = c_{13}^{(j)} \alpha'^{(j)} k^{(j)} + 2\alpha^{(j)}(c_{11}^{(j)} - c_{66}^{(j)})$, $\beta_2^{(j)} = c_{33}^{(j)} \alpha'^{(j)} + 2\alpha^{(j)} c_{13}^{(J)}$, $a_4^{(j)} = \dfrac{k^{(j)}}{k'^{(j)}}$, $\alpha^{(j)}, \alpha'^{(j)}$ are coefficients of temperature extension, k, k' are coefficients of thermal conductivity, $U^{(j)} = (u_1^{(j)}, u_3^{(j)})$ is a displacement vector, $u_4^{(j)}$ is the temperature of body.

We introduce the following definition:

Definition. A vector-function $U(u_1, u_3, u_4)$ defined in $D^{(1)}(D^{(0)})$ is called regular if it has integrable in $D^{(1)}(D^{(0)})$ continuous second derivatives and $U(x)$ itself and its first derivatives are continuously extendable at every point of S and the conditions of infinite are added

$$U(x) = O(1), \ \ \frac{\partial u_k}{\partial x_j} = O(|x|^{-2}), \ \ |x|^2 = x_1^2 + x_3^2, \ \ j = 1, 3, \ k = 1, 3, 4.$$

The stress vector. Throughout this paper $n(x) = n(0, 1)$. Denoting the stress vector by $P(\partial x, n)U$ we have [2]

$$P^{(j)}(\partial x)U^{(j)} = T^{(j)}(\partial x)U^{(j)} - B^{(j)} n u_4^{(j)},$$

where

$$T^{(j)}(\partial x)U^{(j)} = \begin{pmatrix} c_{44}^{(j)}\dfrac{\partial}{\partial x_3} & c_{44}^{(j)}\dfrac{\partial}{\partial x_1}, \\ c_{13}^{(j)}\dfrac{\partial}{\partial x_1} & c_{33}^{(j)}\dfrac{\partial}{\partial x_3} \end{pmatrix} U^{(j)}.$$

For the equation (1), (2) we pose the following contact problem: find a regular solution $U^{(j)}, u_4^{(j)}$ of the equation (1), (2) in the domains $D^{(j)}$, if on the boundary S the following contact conditions are given:

$$\begin{gathered} [U^{(1)}]^+ - [U^{(0)}]^- = f, \ \ [P^{(1)}(\partial x)U^{(1)}]^+ - [P^{(0)}(\partial x)U^{(0)}]^- = F, \\ (u_4^{(1)})^+ - (u_4^{(0)})^- = f_3, \ \ \left[\frac{\partial u_4^{(1)}}{\partial x_3}\right]^+ - \left[\frac{\partial u_4^{(0)}}{\partial x_3}\right]^- = f_4, \end{gathered}$$

where f , f_3 F, f_4 are given function.

From the equation (2) we find $u_4^{(j)}$ and the solution of the equation (1) will be presented in the form $U^{(j)}(x) = V^{(j)}(x) + U_0^{(j)}(x)$, where $V^{(j)}$ is a solution of homogeneous equation $C^{(j)}(\partial x)V^{(j)} = 0$, and $U_0(x)$ is a particular solution of equation (1).

By elementary calculations we obtain the solution $u_4^{(j)}$ in the form

$$\begin{gathered} u_4^{(j)}(x) = \frac{a\sqrt{a_4^{(1)} a_4^{(0)}}}{\pi\sqrt{a_4^{(j)}}} Im \int_{-\infty}^{+\infty} \frac{f_3(t)dt}{t - z_4^{(j)}} - \frac{a}{\pi} Im \int_{-\infty}^{+\infty} i \ln(t - z_4^{(j)}) f_4(t)dt, \\ a = \left[\sqrt{a_4^{(0)}} + \sqrt{a_4^{(1)}}\right]^{-1}, \ \ z_4^{(j)} = x_1 + \alpha_4^{(j)} x_3, \ \ \alpha_4^{(j)} = i\sqrt{a_4^{(j)}}. \end{gathered} \tag{3}$$

Further we assume that (3) is known, when $x \in D^{(1)}$ or $x \in D^{(0)}$. Substitute the $u_4^{(j)}$ in (1) and search the particular solution of the following equation

$$C^{(j)}(\partial x)U^{(j)} = B^{(j)} grad u_4^{(j)}. \tag{4}$$

It easy to check that

$$U_0^{(j)} = \frac{1}{\pi} Im \left[\int_{-\infty}^{+\infty} A^{(j)} \sigma_4^{(j)} grad \sigma_4^{(j)} \ln \sigma_4^{(j)} f_3(t)dt + \int_{-\infty}^{+\infty} D^{(j)} grad \sigma_4^{(j)2} \ln \sigma_4^{(j)} f_4(t)dt\right],$$

is a particular solution of the equation (1), where $\sigma_4^{(j)} = t - z_4^{(j)}$,

$$
\begin{aligned}
&A^{(j)} = \|A_{pq}^{(j)}\|_{2x2}, \;\; A_{11}^{(j)} = A_4^{(j)}, \;\; A_{22}^{(j)} = B_4^{(j)}, \;\; A_{12}^{(j)} = A_{21}^{(j)} = 0, \\
&D^{(j)} = \|D_{pq}^{(j)}\|_{2x2}, \;\; D_{11}^{(j)} = C_4^{(j)}, \;\; D_{22}^{(j)} = D_4^{(j)}, \;\; D_{12}^{(j)} = D_{21}^{(j)} = 0, \\
&A_4^{(j)} = \frac{a\sqrt{a_4^{(1)}a_4^{(0)}}}{\sqrt{a_4^{(j)}}} \frac{\beta_1^{(j)}(c_{44}^{(j)} - c_{33}^{(j)}a_4^{(j)}) + \beta_2^{(j)}a_4^{(j)}(c_{13}^{(j)} + c_{44}^{(j)})}{c_{44}^{(j)}c_{33}^{(j)}(a_4^{(j)} - a_2^{(j)})(a_4^{(j)} - a_3^{(j)})} \\
&B_4^{(j)} = \frac{a\sqrt{a_4^{(1)}a_4^{(0)}}}{\sqrt{a_4^{(j)}}} \frac{\beta_2^{(j)}(c_{11}^{(j)} - c_{44}^{(j)}a_4^{(j)}) - \beta_1^{(j)}(c_{13}^{(j)} + c_{44}^{(j)})}{c_{44}^{(j)}c_{33}^{(j)}(a_4^{(j)} - a_2^{(j)})(a_4^{(j)} - a_3^{(j)})}, \\
&A_4^{(j)} = -2\frac{\sqrt{a_4^{(1)}a_4^{(0)}}}{\sqrt{a_4^{(j)}}}C_4^{(j)}, \quad B_4^{(j)} = -2\frac{\sqrt{a_4^{(1)}a_4^{(0)}}}{\sqrt{a_4^{(j)}}}D_4^{(j)},
\end{aligned}
$$

and the equation (4) takes the form

$$C^{(j)}(\partial x)U^{(j)} = 0.$$

This equation is the equation of the transversally-isotropic elastic body. i.e.. we reduce the solution of basic contact problem of thermoelasticity to the solution of basic contact problem for the equation of a transversally elastic body, whose solution we seek in the form

$$U^{(j)} = \frac{1}{\pi} Im \int_{-\infty}^{+\infty} \sum_{k=2}^{3} N^{(j)(k)} \frac{g^{(j)}(t)dt}{t - z_k^{(j)}}, \;\; z_k^{(j)} = x_1 + \alpha_k^{(j)} x_3,$$

where $g^j(t)$ is unknown density,

$$
\begin{aligned}
&N_{11}^{(j)(k)} = (-1)^k b^{(j)}(c_{33}^{(j)}a_k^{(j)} - c_{44}^{(j)})\frac{\sqrt{a_2^{(1)}a_3^{(0)}}}{\sqrt{a_k^{(j)}}}, \;\; j = 0, 1, \\
&N_{21}^{(j)(k)} = (-1)^k ib^{(j)}(c_{13}^{(j)} + c_{44}^{(j)})\sqrt{a_2^{(j)}a_3^{(j)}} = \sqrt{a_2^{(j)}a_3^{(j)}}N_{12}^{(j)(k)}, \\
&N_{22}^{(j)(k)} = (-1)^k b^{(j)}(c_{44}^{(j)}a_k^{(j)} - c_{11}^{(j)})\frac{1}{\sqrt{a_k^{(j)}}}, \\
&b^{(j)} = \left(\sqrt{a_2^{(j)}} - \sqrt{a_3^{(j)}}\right)^{-1}\left(c_{33}^{(j)}\sqrt{a_2^{(j)}a_3^{(j)}} + c_{44}^{(j)}\right)^{-1},
\end{aligned}
$$

$\alpha_k^{(j)} = i\sqrt{a_k^{(j)}}$, $a_k^{(j)}$, $k = 2, 3$, are roots of characteristic equation

$$(c_{44}^{(j)}a_k^{(j)} - c_{11}^{(j)})(c_{33}^{(j)}a_k^{(j)} - c_{44}^{(j)}) + a_k^{(j)}(c_{13}^{(j)} + c_{44}^{(j)})^2 = 0.$$

Taking into account the boundary condition , for the determination of the unknown density $g^{(j)}$, we obtain the following Fredholm integral equation of second kind

$$g^{(1)}(x_1) + g^{(0)}(x_1) = f(x_1), \tag{5}$$

$$H^{(1)}g'^{(1)}(x_1)+H^{(0)}g'^{(0)}(x_1)+h^{(1)}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{g'^{(1)}(t)dt}{t-x_1}-h^{(0)}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{g'^{(0)}(t)dt}{t-x_1}=F(x_1),$$

where

$$h^{(j)}=\begin{pmatrix} m_{11}^{(j)} & 0\\ 0 & m_{22}^{(j)}\end{pmatrix},\quad H^{(j)}=\begin{pmatrix} 0 & m_{12}^{(j)}\\ m_{21}^{(j)} & 0\end{pmatrix}, j=0,1,$$

$$m_{11}^{(j)}=\frac{c_{44}^{(j)}c_{33}^{(j)}\sqrt{a_2^{(j)}a_3^{(j)}}(\sqrt{a_2^{(j)}}+\sqrt{a_3^{(j)}})}{c_{33}^{(j)}\sqrt{a_2^{(j)}a_3^{(j)}}+c_{44}^{(j)}}=\sqrt{a_2^{(j)}a_3^{(j)}}m_{22}^{(j)}>0,$$
$$m_{12}^{(j)}=\frac{c_{44}^{(j)}(c_{33}^{(j)}\sqrt{a_2^{(j)}a_3^{(j)}}-c_{13}^{(j)})}{c_{33}^{(j)}\sqrt{a_2^{(j)}a_3^{(j)}}+c_{44}^{(j)}}=-m_{21}^{(j)},\quad m_{12}^{(j)}>0.$$

Rewriting the equation (5) in the form

$$\begin{aligned}&[H^{(0)}-H^{(1)}]g^{(0)}-(h^{(0)}+h^{(1)})\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{g^{(0)}(t)dt}{t-x_1}\\&=c+\widehat{F}-H^{(1)}f(x_1)-h^{(1)})\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(t)dt}{t-x_1},\end{aligned}\tag{6}$$

where c is an arbitrary constant, $\widehat{F}=\int\limits_0^{x_1}F(t)dt$.

Using the permutation formula

$$\frac{1}{\pi^2}\int_{-\infty}^{+\infty}\frac{dx_1}{x_1-\xi}\int_{-\infty}^{+\infty}\frac{g(t)dt}{t-x_1}=-g(\xi).$$

From (6) it follows

$$\begin{aligned}&[h^{(0)}+h^{(1)}]g^{(0)}+(H^{(0)}-H^{(1)})\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{g^{(0)}(t)dt}{t-x_1}\\&=ci+\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\widehat{F}(t)dt}{t-\xi}+h^{(1)}f(\xi)-H^{(1)})\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(t)dt}{t-\xi}.\end{aligned}\tag{7}$$

Let's introduce the following notations:

$$g^{(0)}+\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{g^{(0)}(t)dt}{t-\xi}=G(\xi),\quad \widehat{F}+\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{\widehat{F}(t)dt}{t-\xi}=\varphi(\xi),$$

$$f+\frac{i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{f(t)dt}{t-\xi}=\widehat{f}(\xi).$$

Then the equations (6), (7) can be rewritten in the form of one equation

$$DG=[H^{(0)}-H^{(1)}+i(h^{(0)}+h^{(1)})]G=\varphi+(ih^{(1)}-H^{(1)})\widehat{f}. \tag{8}$$

By the direct calculations we get

$$detD=\Delta=-[A^{(0)}+A^{(1)}+m_{11}^{(0)}m_{22}^{(1)}+m_{11}^{(1)}m_{22}^{(0)}+2m_{12}^{(0)}m_{12}^{(1)}]<0,$$

where

$$A^{(j)}=\frac{c_{44}^{(j)}(c_{11}^{(j)}c_{33}^{(j)}-c_{13}^{(j)2})}{(c_{33}^{(j)}\sqrt{a_2^{(j)}a_3^{(j)}}+c_{44}^{(j)}}>0.$$

It can be easily seen that the determinant of system (8) is not zero and from (8) we can determine G. Equating the real parts of both sides, we find

$$g^{(j)}=(-1)^{j+1}Q\widehat{F}+\left[\delta_{1j}+(-1)^j(QH^{(1)}-qh^{(1)})\right]f$$
$$+\frac{(-1)^{j+1}}{\pi}q\int\limits_{-\infty}^{+\infty}\frac{\widehat{F}dt}{t-\xi}-\frac{(-1)^j}{\pi}\left(qH^{(1)}+Qh^{(1)}\right)\int\limits_{-\infty}^{+\infty}\frac{f(t)dt}{t-\xi}.$$

Substituting the densities $g^{(j)}$ obtained, into (8) and taking into account the following formula [1]

$$\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\frac{dt}{(t-z_k^{(j)})(\xi-t)}=\frac{(-1)^{j+1}i}{\xi-z_k^{(j)}},\quad \frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\frac{dt}{(\xi-z_k^{(j)})}=(-1)^{j+1}i,$$

we obtain the solution of formulated problem in quadratures

$$U^{(j)}=\frac{1}{\pi}Im\sum_{k=2}^{3}N^{(j)(k)}\left[M^{(j)}\int\limits_{-\infty}^{+\infty}\frac{f(t)dt}{t-z_k^{(j)}}\right.$$
$$\left.+R^{(j)}\int\limits_{-\infty}^{+\infty}F(t)\ln(t-z_k^{(j)})dt\right],$$

where

$$R^{(j)}=-iq+(-1)^jQ\ \ ,M^{(j)}=\delta_{1j}E+(-1)^j(QH^{(1)}-qh^{(1)})-i(qH^{(1)}+Qh^{(1)}),$$
$$q=\frac{1}{\Delta}\begin{pmatrix} m_{22}^{(0)}+m_{22}^{(1)} & 0\\ 0 & m_{11}^{(0)}+m_{11}^{(1)}\end{pmatrix},\quad Q=\frac{1}{\Delta}\begin{pmatrix} 0 & m_{12}^{(0)}-m_{12}^{(1)}\\ m_{21}^{(0)}-m_{21}^{(1)} & 0\end{pmatrix}.$$

Thus we have obtained the Poisson formula for the solution of the basic contact problem for piece-wise homogeneous plane.

For the solution to be regular in $D^{(j)}$ it's sufficient that the functions satisfy Holder's conditions $f, f_3 \in C^{1,\alpha}(S), F, f_4 \in C^{0,\alpha}(S)$, to be integrable absolutely on the whole axis and to have the following order at infinity

$$F, f_4 = O(|t|^{-1-\beta}), \quad f, f_3 = O(|t|^{-\beta}), \quad \beta > 0.$$

In additional for the functions $U^{(j)}, \;\; u_4^{(j)}$ to be single valued and bounded at infinite, we assume that the conditions $\int\limits_{-\infty}^{+\infty} F(t)dt = 0$ and $\int\limits_{-\infty}^{+\infty} f_4(t)dt = 0$ are fulfilled.

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