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ON THE DOMAINS OF PROPAGATION OF WAVES DETERMINED BY PERTURBATIONS ON THE CLOSED SUPPORT

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Abstract. The Cauchy problem with closed support of data for quasi-linear hyperbolic equation with the possible parabolic degeneracy is investigated. The structure of the domain of the solution is studied. Initial conditions stimulating the formation of areas impenetrable for non-linear waves are founded.

Keywords and phrases: Quasi-linear hyperbolic equation, parabolic degeneracy, Cauchy problem, closed support.

AMS subject classification: 35L72, 35L15.

It is known, that Cauchy problem with closed support is not generally correct for hyperbolic equations. However, according to J. Hadamard [1], this kind of problem may be set correctly in some cases. Such problems for some linear equations were studied by S.L. Sobolev [2], R.A. Alexandryan [3], N.N. Vakhania [4], A.M. Nakhushev [5], etc. Differing from them we managed to find exact area of definition of the solution in non-linear case.

We investigated the class of second order quasi-linear equations of hyperbolic type with possible parabolic degeneracy. The breaking of hyperbolicity can be expressed as Tricomian or Cibrario-Keldysh type parabolic degeneracy. The principle coefficients of these equations are second power polynomials of its lower order derivatives. In such a situation, the families of characteristics and domains of definition of solutions are not known from the beginning and should be determined simultaneously with the sought-for solution. From this point, the existence of representations of general integrals of quasi-linear equations appears quite useful for investigation of such problems.

We consider the following equation:

$$u_y(u_y-1)u_{xx}+(u_y-u_x-2u_xu_y+1)u_{xy}+u_x(u_x+1)u_{yy}=0. \tag{1}$$

It's a hyperbolic equation with possible parabolic degeneracy and the set of hyperbolic solutions is described by relation

$$u_x-u_y+1\neq 0.$$

Characteristic invariants are described by the following systems:

$$\begin{cases}\xi_1=u+x,\\ \xi_2=\dfrac{u_x}{u_x-u_y+1}.\end{cases}\qquad \begin{cases}\eta_1=u-y,\\ \eta_2=\dfrac{u_x+1}{u_x-u_y+1}.\end{cases}$$

And the General integral of the equation (1) is the sum of two arbitrary functions $f,g\in C^2(R)$:

$$f(u+x)+g(u-y)=y \tag{2}$$

For the equation (1) we studied the Cauchy problem posed on the circle:

$$\gamma: \quad (x-a)^2+y^2=(\sqrt{2}-a)^2, \qquad a<0.$$

Using polar coordinates

$$x=\rho\cos t, \quad y=\rho\sin t,$$

we can write this problem as follows:

$$\begin{gathered} u|_\gamma=\tau(t), \quad t\in[0,2\pi], \\ u_\rho|_\gamma=\nu(t), \quad t\in[0,2\pi],\ \nu\in C^2[0.2\pi], \end{gathered} \tag{3}$$

where

$$\gamma: \quad \rho=a\cos t+\sqrt{a^2\cos^2 t+2-2\sqrt{2}a}, \quad t\in[0,2\pi]. \tag{4}$$

If we subject the General integral (2) to initial conditions (3), we obtain the implicit solution for the problem (1), (3):

$$\int_{T_2(u-y)}^{T_1(u+x)} H(t)\,dt+(2-a)\sin(T_2(u-y))=y, \tag{5}$$

where

$$H(t)=\frac{(2-a)\left[((a-1)\sin t-\nu(t))\cos t+\tau'(t)\sin t\right]}{((a-2)\nu(t)+\tau'(t))\cos t+((2-a)\nu(t)+\tau'(t))\sin t+a-2}\left[\tau'(t)+(a-2)\sin t\right],$$

$$\begin{gathered} z=\tau(t)+(2-a)\cos t+a, \quad t=T_1(z), \\ w=\tau(t)+(2-a)\sin t, \quad t=T_2(w). \end{gathered}$$

On the base of (5) it's easy to describe the both of families characteristic curves. They are one-parameter families, which come out from arbitrary point of support (4):

$$y=\int_{T_2(c-y-x)}^{c} H(t)\,dt+(2-a)\sin\left(T_2(c-y-x)\right), \tag{6}$$

$$y=\int_{c}^{T_1(c+y+x)} H(t)\,dt+(2-a)\sin\left(T_2(c)\right). \tag{7}$$

As it is known the discriminant curve for the family (6) exists when the following system is solvable in regard to variables x and y:

$$y=\int_{T_2(c-y-x)}^{c} H(t)\,dt+(2-a)\sin\left(T_2(c-y-x)\right),$$

$$H(c)-H\left(T_2(c-y-x)\right)T_2'(c-y-x)+(2-a)\cos\left(T_2(c-y-x)\right)T_2'(c-y-x)=0.$$

Where the second equation of the system is obtained by derivation of (6) with respect to parameter c. Analogously, we can write the similar system in the case of family (7):

$$y=\int_{c}^{T_1(c+y+x)} H(t)\,dt+(2-a)\sin\left(T_2(c)\right),$$

$$H\left(T_1(c+y+x)\right)T_1'(c+y+x)-H(c)+(2-a)\cos\left(T_2(c)\right)T_2'(c)=0.$$

Among multiple variants of concrete initial conditions, we have found an example when for the families (6) and (7) discriminant curve is the circle line:

$$\delta: \quad x^2+y^2=1. \tag{8}$$

These concrete initial conditions are the following:

$$u|_\gamma=u_0+\left(a\cos t+\sqrt{a^2\cos^2 t+2-2\sqrt{2}a}\right)\sin t, \quad t\in[0,2\pi],$$

$$u_\rho|_\gamma=\sin t, \quad t\in[0,2\pi],$$

where u_0 is the value of the solution in arbitrary point of the support, and the characteristic curves are straight lines

$$x\cos t+y\sin t=1, \quad t\in[0,2\pi].$$

Each of these straight lines touches the circle (8) at one point. This point divides each straight line into two rays. Each of these rays belongs to different families of characteristics. Each line of one family of characteristics crosses the line of other family only one time. So, the question of correctness of the Cauchy problem in our case is fixed. None of the straight lines penetrate inside the circle area. So, we found exactly the area of definition of the solution of the problem (1), (3) posed on the closed curve (4). The solution is defined everywhere except the inside area of the circle (8). At the same time the circle (8) is the envelope for the both families of characteristics. Thus we have the parabolic degeneration on the line of the circle. So, in this concrete case we have proved the following theorem

Theorem. *Cauchy problem (1), (3) posed on the closed curve (4) is correct and the solution of the problem is defined everywhere except the inside area of the circle (8).*

R E F E R E N C E S

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