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ON ONE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION

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Abstract. One nonlinear integro-differential equation is considered. Large time behavior of solution as $t \to \infty$ is studied. The finite difference scheme is investigated as well. Results of the numerical experiments are given.

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In this note one nonlinear integro-differential equation is considered. This equation is a one-dimensional and one-component analog of the model which is based on the classical Maxwell system [1]. This system arises in the process of penetration of a magnetic field into a substance. To the integro-differential form at first it was reduced in the work [2].

Let us consider the following nonlinear integro-differential equation

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)\frac{\partial U}{\partial x}\right] = 0, \tag{1}$$

where $a = a(S) \geq a_0 = Const > 0$ is a given function of its argument.

Many scientific works are devoted to the investigation and numerical resolution of (1) type models. The existence and uniqueness of the solution of the initial-boundary value problems for (1) kind models in suitable classes have been proved in [2]-[9] and in a number of other works as well. Asymptotic behavior of solutions as $t \to \infty$ is investigated in many works also (see, for example, [8],[10]-[12] and references there in). Numerical resolution by finite difference schemes is given in the works [10],[11],[13]-[15].

The aim of this note is to study asymptotic behavior of solution and to construct approximate solutions for the initial-boundary value problem of the equation (1) with source term. The considered equation has the form

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)\frac{\partial U}{\partial x}\right] + |U|^{q-2}U = 0, \tag{2}$$

where $q \geq 2$.

In the domain $[0,1] \times [0,\infty)$ let us consider the following initial-boundary value problem:

$$\begin{gathered} U(0,t) = U(1,t) = 0, \\ U(x,0) = U_0(x), \end{gathered} \tag{3}$$

where $U_0 = U_0(x)$ is a given function.

It is not difficult to get the following statement.

Theorem 1. *If $a(S) \geq a_0 = Const > 0$, $q \geq 2$, $U_0 \in L_2(0,1)$ then problem (2),(3) has not more than one solution and the following asymptotic property takes place*

$$\|U(x,t)\| \leq Ce^{-a_0 t}.$$

Here $\|\cdot\|$ is the usual norm of the space $L_2(0,1)$.

Now let us construct the finite difference scheme for the problem (2),(3). At first in the rectangle $[0,1] \times [0,T]$ let us introduce uniform grid with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0,1,...,M$; $j = 0,1,...,N$ with $h = 1/M$, $\tau = T/N$. The initial line is denoted by $j = 0$. The discrete approximation at (x_i, t_j) is designed by u_i^j and the exact solution to the problem (2),(3) by U_i^j.

Using usual notations and the methods of construction of difference schemes (see, for example, [16]) let us construct following finite difference scheme for the problem (2),(3):

$$\frac{u_i^{j+1} - u_i^j}{\tau} - \left\{ a\left(\tau \sum_{k=1}^{j+1} \left(u_{\bar{x},i}^k\right)^2\right) u_{\bar{x},i}^{j+1} \right\}_x + |u_i^{j+1}|^{q-2} u_i^{j+1} = 0, \quad (4)$$
$$i = 1,2,...M-1; \;\; j = 0,1...N-1,$$

$$u_0^j = u_M^j = 0, \quad j = 0,1...,N, \quad (5)$$

$$u_i^0 = U_{0,i}, \quad i = 0,1...,M. \quad (6)$$

In [10] convergence of the scheme (4)-(6) without source term for the case $a(S) = 1 + S$ is considered.

The following statement takes place.

Theorem 2. *If $a(S) = 1 + S$, $q \geq 2$ and the initial-boundary value problem (2),(3) has the sufficiently smooth solution $U = U(x,t)$ then the finite difference scheme (4)-(6) converges to the solution of problem (2),(3) and the following estimate is true*

$$\left\|u^j - U^j\right\| \leq C(\tau + h).$$

Here $\|\cdot\|$ is a discrete analog of the norm of the space $L_2(0,1)$ and C is a positive constant independent of τ and h.

Note that for solving the finite difference scheme (4)-(6) Newton's iterative process [17] is used and great number of numerical experiments are performed. These experiments agree with theoretical investigations.

The test given on the figures below (see, Fig. 1) has the form $U(x,t) = x(1 - x)\sin(x+t)$.

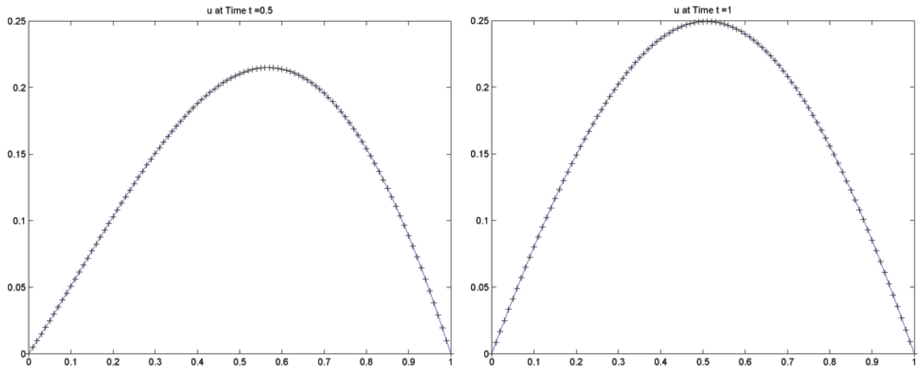

Fig. 1. The solutions at t = 0.5 and t=1. The exact solution is solid line and the numerical solution is marked by *.

R E F E R E N C E S

1. Landau L., Lifschitz E. Electrodynamics of Continuous Media. (Russian) *Moscow*, 1958.
2. Gordeziani D.G., Dzhangveladze T.A., Korshia T.K. Existence and uniqueness of the solution of a class of nonlinear parabolic problems. *Differ. Uravn.*, **19**, 7 (1983), 1197-1207 (Russian). English translation: *Differ. Equ.*, **19**, (1984), 887-895.
3. Dzhangveladze T.A. First boundary-value problem for a nonlinear equation of parabolic type. (Russian) *Dokl. Akad. Nauk SSSR*, **269**, 4 (1983), 839-842. English translation: *Soviet Phys. Dokl.*, **28**, 4 (1983), 323-324.
4. Dzhangveladze T.A. Issledovanie Pervoi Kraevoi Zadachi Dlya Nekotorykh Nelineinykh Integro-differential'nykh Iravnenii Parabolicheskogo Tipa. (Investigation of the First Boundary Value Problem for Some Nonlinear Integro-differential Equations of Parabolic Type). (Russian) *Tbilisi: Tbilis. Gos. Univ.*, 1983.
5. Laptev G.I. Quasilinear parabolic equations which contains in coefficients Volterra's operator. (Russian) *Math. Sbornik,* **136** (1988), 530-545. English translation: *Sbornik Math.,* **64** (1989), 527-542.
6. Lin Y., Yin H.M. Nonlinear parabolic equations with nonlinear functionals. *J. Math. Anal. Appl.*, **168**, 1 (1992), 28-41.
7. Long N.T., Dinh A.P.N. Nonlinear parabolic problem associated with the penetration of a magnetic field into a substance. *Math. Meth. Appl. Sci.,* **16** (1993), 281-295.
8. Jangveladze T. On one class of nonlinear integro-differential equations. *Semin. I.Vekua Inst. Appl. Math. Rep.*, **23** (1997), 51-87.
9. Bai Y., Zhang P. On a class of Volterra nonlinear equations of parabolic type. *Appl. Math. Comp.,* **216** (2010), 236-240.
10. Jangveladze T., Aptsiauri M. The asymptotic behavior of solutions and finite difference scheme for one nonlinear integro-differential equation. *Bull. Georgian Acad. Sci.*, **174**, 1 (2006), 25-28.
11. Dzhangveladze T.A., Kiguradze Z.V. Asymptotic behavior of the solution to nonlinear integro-differential diffusion equation. (Russian) *Differ. Uravn.*, **44**, 4 (2008), 517-529. English translation: *Differ. Equ.*, **44**, 4 (2008), 538-550.
12. Jangveladze T., Kiguradze Z., Neta B. Large time behavior of solutions to a nonlinear integro-differential system. *J. Math. Anal. Appl.*, **351**, 1 (2009), 382-391.
13. Jangveladze T. Convergence of a difference scheme for a nonlinear integro-differential equation. *Proc. I. Vekua Inst. Appl. Math.*, **48**, (1998) 38-43.
14. Jangveladze T., Kiguradze Z., Neta B. Finite difference approximation of a nonlinear integro-differential system. A*ppl. Math. Comput.,* **215**, 2 (2009), 615-628.

15. Kiguradze Z. Finite difference scheme for a nonlinear integro-differential system. *Proc. I. Vekua Inst. Appl. Math.*, **50-51**, (2000-2001), 65-72.

16. Samarskii A.A. The Theory of Difference Schemes. (Russian) *Moscow,* 1977.

17. Rheinboldt W.C. Methods for Solving Systems of Nonlinear Equations. *SIAM, Philadelphia*, 1970.

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