Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 24, 2010

SYSTEMS OF EQUATIONS IN ONE VARIABLE OVER FREE NILPOTENT GROUPS OF NILPOTENCY CLASS 2

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Abstract. A complete classification of algebraic sets and coordinate groups is given for systems of equations in one variable over a free nilpotent group.

Keywords and phrases: Algebraic geometry over a group, algebraic set, coordinate group.

AMS subject classification: 20B07.

Many connections between the subsets of elements of a fixed algebraic system A can be expressed in terms of systems of algebraic equations over A. In the classical case, where A is a field, the division of mathematics which studies connections of such kind is called algebraic geometry. It is natural to extend this notion also to the case of an arbitrary algebraic system A.

Like in the classical case, the main problem of algebraic geometry over A is the problem of classification of algebraic sets, i.e. of sets of solutions of systems of algebraic equations over A. A sufficiently large concrete material of analysis of structures of algebraic sets has presently been accumulated for concrete algebraic systems (groups, rings, Lie algebras and so on) and there has arisen a need in theoretical comprehension of this material.

The basic notions and results of algebraic geometry over groups are expounded in [1], [2]. In the present report we study algebraic geometry over a free nilpotent group G of nilpotency class 2. Namely, we consider the algebraic sets and coordinate groups for the systems of equations in one variable over G. Note that an analogous problem for a free group G we studied in [3]-[6]. The final theorem on the structure of algebraic sets and coordinate groups over a free group was given in [7]. The case of a free metabelian group G was investigated in [3]-[10], and the final results were obtained in [11].

On nilpotent groups of nilpotency class 2. Denote by $\mathfrak{N}_2$ the variety of groups of nilpotency class 2. If G is a group from $\mathfrak{N}_2$ then its commutant G' lies in the $Z(G)$–centre of G.

Let now G be a free nilpotent group of rank $r > 1$, $G = \langle a_1, \dots, a_r\rangle$, where $A = \{a_1, \dots, a_r\}$ is a system of free generators for G. Denote by $c_{ji} = [a_j, a_i]$, where $j > i$, the basic commutators of weight 2 constructed on the set A. It is well known (see e.g. [12], [Proposition 3.1]) that an arbitrary element $g \in G$ has a representation of the form

$$g = a_1^{\alpha_1} \cdots a_r^{\alpha_r} \prod c_{ji}^{\beta_{ji}}, \tag{1}$$

where $\alpha_i \in \mathbb{Z}$, $\beta_{ji} \in \mathbb{Z}$, and this representation is unique. Furthermore, it is well known that $Z(G) = G'$. Let us also present some other available results on the group G:

- an element g of the form (1) is primitive for G (i.e. it can be included in a system of free generators for G) if and only if the row $(\alpha_1, \dots, \alpha_r)$ is unimodular;

- if $g \notin Z(G)$, then its centralizer $C_G(g)$ is an abelian subgroup, and if $g = a_1^{\alpha_1' d} \cdots a_r^{\alpha_r' d} b$, $d = gcd(\alpha_1, \ldots, \alpha_r)$, and the row $(\alpha_1', \ldots, \alpha_r')$ is unimodular, then $C_G(g) = C_G(g')$, where $g' = a_1^{\alpha_1'} \cdots a_r^{\alpha_r'} \cdots$. Furthermore, if $h \in C_G(g)$ then $h \equiv g'^{\gamma} \pmod{Z(G)}$, $\gamma \in \mathbb{Z}$.

Recall that for every finitely generated nilpotent group G, there is a finite series of normal subgroups with cyclic factors. The number $h(G)$ of infinite and cyclic factors does not depend on the series and is called the **Hirsch number** of G.

Key example. Let K be a commutative ring with unit 1. Then for the group

$$UT_3(K) = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \middle| \text{ the symbol } * \text{ stands for elements } K \right\}.$$

For this group

$$Z\big(UT_3(K)\big) = UT_3(K)' =$$
$$= \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \text{ the symbol } * \text{ stands for elements } K \right\}.$$

Some aspects of algebraic geometry over groups. The basic notions and results of algebraic geometry over groups are presented in [1], [2]. For completeness, we formulate some of them, especially for the case of nilpotent groups of nilpotency class 2 [12]. Let G be a group in $\mathfrak{N}_2$. The Cartesian product $G^n = G \times \cdots \times G$ (n copies) is called **an affine space over** G. Let $X = \{x_1, \ldots, x_n\}$ be a set of letters, and let $G[X]$ denote a nilpotent product $G \underset{\mathfrak{N}_2}{*} F(X)$, where $F(X)$ is a free nilpotent group in $\mathfrak{N}_2$ with base X. A system of equations S over is a subset of $G[X]$. An element $u \in S$ can be considered as a non-commuting polynomial $u = u(x_1, \ldots, x_n)$ in variables $x_1, \ldots, x_n$ with coefficients in G. An element $p = (g_1, \ldots, g_n) \in G^n$ is called a **root** of $u = u(x_1, \ldots, x_n)$ if $u\langle g_1, \ldots, g_n \rangle = 1$ in G. Given a subset S of $G[X]$, p is called a **root** of S if p is a root of every $u \in S$.

Definition 1. A subset V of an affine space G^n is called an **algebraic set** over G if V is the set of all solutions to a system of equations $S \subseteq G[X]$.

Given S, we denote by $V_G(S)$ the algebraic set of all solutions to the system S. Furthermore, for V and S such that $V = V_G(S)$, define

$$Rad(V) = \Big\{ u \in G[X] \big| \ u(p) = 1 \text{ for every } p \in V_G(S) \Big\}.$$

It is clear that $Rad(V)$ is always a normal subgroup of $G[X]$.

Definition 2. The group $\Gamma(V) = G[X]/Rad(V)$ is called the **coordinate group** of an algebraic set V.

Furthermore, considering all algebraic sets from G^n as a pre-base of closed sets, we transform G^n into a topological space (the Zariski topology). In a standard manner, we define the notion of an irreducible algebraic set of G^n. The coordinate group of an irreducible algebraic set is called an **irreducible coordinate group**. As is known [1], the coordinate group of an algebraic set over G is a G-subgroup of the Cartesian product $G^I = \prod\limits_{i\in I} G^{(i)}$, where $G^{(i)} \cong G$, $i \in I$, and G can be identified with the diagonal of the group G^I, $\Delta : G \to G^I$, $\Delta(g) = (\dots, g, \dots)$. In what follows, we consider only the systems of equations in one variable and the algebraic sets of G.

Description of coordinate groups and algebraic sets. In this section, we give a complete classification of the coordinate groups for the systems of equations in one variable over a free nilpotent group of nilpotency class 2.

The following important result is not true for an arbitrary group, but is true for finitely generated nilpotent groups.

Lemma. *Let G be a finitely generated nilpotent group, and let H be the coordinate group of an algebraic set over G. Then there is a natural number k such that H is a G-subgroup of $G^k = \underbrace{G \times \cdots \times G}_{k\text{-times}}$, and G is diagonally embedded in G^k.*

Using this lemma, in the next theorem we give a complete description of coordinate groups for systems of equations in one variable over a free nilpotent group of class 2.

Theorem 1. *Let $H = \langle \Delta(G), x\rangle$, $x = (g_1, \dots, g_k)$. Then one of the following holds:*

1. *$x \in \Delta(G)$ and then $H \cong G$;*
2. *up to translation by an element $\Delta(g) = (g, \dots, g)$, the element x is such that $g_i \in Z(G)$ and $x \notin \Delta(G)$, and in this case $H = G \times \langle x\rangle$;*
3. *up to translation by an element $\Delta(g) = (g, \dots, g)$, the element x is such that $fx \notin Z(G^k)$ for all $f \in \Delta(G)$ and there is an element $g \notin Z(G)$ such that $g_i \in C_G(g)$; in this case $H = \langle G, x | \ [x, g_0] = 1\rangle_{\mathfrak{N}_2}$, where $g = g_0^l c$, $c \in Z(G)$, $l \in \mathbb{Z}$ and g_0 is not a square modulo $Z(G)$ (g_0 is a root element);*
4. *if none of the conditions 1–3 holds then $H = G \underset{\mathfrak{N}_2}{*} \langle x\rangle$ is a free nilpotent group in $\mathfrak{N}_2$ of rank $r + 1$.*

Using Theorem 1, we obtain a complete classification of algebraic sets and coordinate groups for a free nilpotent group of class 2.

Theorem 2. *Each coordinate group H for a system of equations in one variable is an irreducible coordinate group.*

Theorem 3. *Each algebraic set over a free nilpotent group G of rank $r > 1$ in $\mathfrak{N}_2$ is, up to an isomorphism, one of the following:*

1. *a point;*
2. *the center $Z(G)$ of G;*
3. *the centralizer of an element $g \in G$, $g \notin Z(G)$;*
4. *the whole group G.*

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Received 18.04.2010; revised 15.09.2010; accepted 17.10.2010.

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