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SOME REMARKS ON GENERALIZED BELTRAMI SYSTEMS

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Abstract. The generalized Beltrami systems in the complex plane are considered and the modified Dirichlet problem for such system is solved.

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The theory of elliptic systems on the plane is the classical object of investigation. We consider the first order system of partial differential equations in the complex plane C of the following form

$$w_{\overline{z}} = Q\, w_z, \tag{1}$$

where Q is a given $n \times n$ complex matrix of the class $W_p^1(C)$, $p > 2$ and $Q(z) = 0$ outside of some circle, satisfying the condition

$$Q(z_1)\, Q(z_2) = Q(z_2)\, Q(z_1). \tag{2}$$

Hile [1] notices that what appears to be the essential property of the elliptic systems in the plane for which one can obtain a useful extension of the analytic function theory is the self commuting property of the variable matrix Q, which is the condition (2), for any two points z_1, z_2 of the complex plane. Further, such a matrix cannot be brought into the quasi-diagonal form of Bojarski [2] by a similarity transformation. So, Hile [1] attempts to extend the results of Bojarski to a wider class of the systems in the some form as (1).

Following Hile if Q is self commuting in C, and if $Q(z)$ has eigenvalues less than 1, then system (1) is called generalized Beltrami system. Solution of such system will be called Q-holomorphic vector. Under a solution of the equation (1) in some domain D we mean so-called regular solution [3].

The matrix valued function $\Phi(z)$ is a generating solution of the system (1) if it satisfies the following properties [1]:

(i) $\Phi(z)$ is a C^1-solution of (1) in C;

(ii) $\Phi(z)$ is self-commuting and commutes with Q in C;

(iii) $\Phi(t) - \Phi(z)$ is invertible for all z, t in C, $z \neq t$;

(iv) $\partial_z \Phi(z)$ is invertible for all z in C.

The matrix $V(t,z) = \partial_t\, \Phi(t) \big[\Phi(t) - \Phi(z)\big]^{-1}$ we call the generalized Cauchy kernel for the system (1).

Introduce some classes of Q-holomorphic vectors. Let D be a bounded domain with sufficiently smooth boundary. We say, that Q-holomorphic vector $\Phi(z)$ belongs to the class $E_p(D,Q)$, $p>1$, if the vector $\Phi(z)$ is representable in the domain D by the generalized Cauchy-Lebesgue type integral

$$\Phi(z)=\frac{1}{2\pi i}\int_\Gamma V(t,z)\,d_Q\,t\,\varphi(t), \tag{3}$$

where $d_Qt=I\,dt+Q\,d\bar t$, I is a identity matrix.

We need the following result in the sequel (see [4]).

Theorem 1. *Let $\Phi(z)\in L_p(\Gamma)$, $p>1$. Then the generalized Cauchy-Lebesgue integral* (3) *belongs to the class $L_s(\overline{D})$, $s=2p$, moreover, the following inequality*

$$\big\|\Phi\big\|_{L^n_s(\overline{D})}\le M(p,D)\big\|\varphi\big\|_{L^n_p(\Gamma)}$$

holds. (*The notation $A\in K$, where A is a matrix and K is some class of functions, means that every element $A_{\alpha\beta}$ of A belongs to K. If K is some linear normed space with the norm $\{\cdot\}_K$, then $\|A\|_K=\max\limits_{\alpha,\beta}\|A_{\alpha\beta}\|_K$.*)

Consider now the boundary value problem for the system (1), the so-called modified Dirichlet problem (see [5])

$$\operatorname{Re}\big[w(t)\big]=f(t)+a(t),\quad t\in\Gamma, \tag{4}$$

$$\operatorname{Im}\big[w(z_0)\big]=c_0,\qquad z_0\in D, \tag{5}$$

where $f(t)$ is a given real vector on Γ, c_0 given real constant vector, z_0 is an arbitrary fixed point of the domain D, $a(t)=aj$ on Γ, $j=0,\dots,m$ - piecewise continuous vector not defined beforehand. The vector $a(t)$ is completely defined by the problem itself, if one of its values is arbitrarily fixed. We assume, that $a_0=0$ in what follows. In case of absence of the curves $\Gamma_1,\dots,\Gamma_m$ we get the conditions of ordinary Dirichlet problem:

Every Q-holomorphic vector $\Phi(z)$ of the class $E_p(D,Q)$ admits the following representation:

$$\Phi(z)=\frac{1}{\pi i}\int_\Gamma V(t,z)\,d_Q\,t\,\mu(t)+iC, \tag{6}$$

where $\mu(t)\in L_p(\Gamma,\rho)$ is a real vector. The vector $\mu(t)$ is defined on Γ_j, $j\ge1$, uniquely to within the constant vector, $\mu(t)$ on Γ_0 and the constant vector C is defined by the vector $\Phi(z)$ uniquely.

Using the representation (6) and the analog of the Sokhotsky-Plemelj formula for the integral (6) and the boundary condition (4) we obtain the system of Fredholm type integral equations:

$$\big[(I+M)\mu\big](t)=f(t)+a(t). \tag{7}$$

The corresponding homogeneous equation of (7) has nm linearly independent solutions, therefore, the inhomogeneous equation (7) is solvable if and only if the right-hand side of the equation satisfy some nm conditions. For the solution of the problem the constant vectors a_j should be chosen such, that these conditions will be fulfilled.

Following [5] instead of the equation (7) consider its equivalent equation:

$$(K\mu)(t_0)\equiv\big[(I+M)\mu\big](t_0)-\int_\Gamma k(t,t_0)\,\mu(t)\,ds=f(t_0), \tag{8}$$

where $k(t, t_0)$ is a diagonal matrix:

$$k(t, t_0) = \begin{cases} \rho_j(t)\, I, & t_0, t \in \Gamma_j, \quad j = 1, \dots, m, \\ 0, & \text{in all other cases.} \end{cases}$$

$\rho_j(t)$ denotes real continuous function given on Γ_j $(j = 1, \dots, m)$ satisfying the condition

$$\int_{\Gamma_j} \rho_j(t)\, ds \neq 0.$$

It can be proved that the homogeneous equation (8) have not the non-trivial solutions. From this by virtue of Fredholm theorem it follows, that inhomogeneous equation (8) has always unique solution $\mu(t)$, which is a solution of initial equation (7). In addition the vectors a_j have completely defined values, namely $a_j = \int\limits_{\Gamma_j} \rho_j \mu ds$.

We have, that the linear bounded operator K, appearing in (8), is invertible operator in Banach space $L^n_p(\Gamma)$, $p > 1$. Therefore,

$$\|\mu\|_{L^n_p(\Gamma)} \leq A(p, \Gamma) \|f\|_{L^n_p(\Gamma)},$$

where

$$A(p, \Gamma) = \|K^{-1}\|_{L^n_p(\Gamma)}.$$

Applying the estimation given in Theorem 1, it is possible to get the following estimation

$$\|\Phi\|_{L^n_s(\overline{D})} \leq B(p, \Gamma) \|f\|_{L^n_p(\Gamma)} + \sum_{i=1}^{n} |c_{0,i}| (mD)^{1/p},$$

where Φ is a Q-holomorphic vector resolving the problem (4), (5), mD is a measure of the domain D.

Consider the nonlinear differential system in the domain D which has the following complex form:

$$\frac{\partial w}{\partial \overline{z}} - Q(z) \frac{\partial w}{\partial z} = F(\cdot\,, w), \tag{9}$$

where $F(\cdot\,, w)$ is a $n \times 1$-matrix.

Investigate the boundary value problem (4), (5) for the system (9), $f(t) \in L_p(\Gamma)$, $p > 1$ is a given vector on Γ, we seek the desired solution $w(z) = (w_1, \dots, w_n)$ in the class $E_p(D, Q) + C(\overline{D})$. More precisely, the boundary value problem (4), (5) in this case is posed in the following form:

Find a vector $w(z) = (w_1, \dots, w_n) \in W^1_p(D)$, $p > 2$, from the class $E_p(D, Q) + C(\overline{D})$, satisfying (9) almost everywhere in D and the condition (4) almost everywhere the on Γ.

With respect to F we suppose to be fulfilled the following conditions:

1) $F(z, w)$ is measurable with respect to z for the fixed w;

2) F satisfies the Lipshitz condition:

$$|F(z, w_1) - F(z, w_2)| \leq L|w_1 - w_2|, \quad L > 0;$$

3) $F(z,0) \in L_s(\overline{D})$, where $s = 2p$.

From these conditions it follows that $F(z,w) \in L_s(\overline{D})$ for every vector $w \in L_s(\overline{D})$.

As a corollary of above obtained results it is possible to show, that the following theorem is valid:

Theorem 2. *If the right-hand side F of the system* (9) *satisfies the restrictions* 1)–3) *then in the case of sufficiently small Lipshitz constant L the boundary value problem* (4), (5) *for every given $f(t) \in L_p(\Gamma)$, $p > 1$ and c_0 has unique solution of the class $E_p(D,Q) + C(\overline{D})$. The piecewise constant vector $a(t)$ is defined completely by the problem itself, if one of its values is arbitrarily fixed.*

REFERENCES

1. Hile G. N. Function theory for generalized Beltrami systems. *Contemp. Math.* **11** (1982), 101–125.

2. Bojarski B. Theory of generalized analytic vectors. (Russian) *Ann. Polon. Math.* **17** (1966), 281–320.

3. Vekua I. Generalized Analytic Functions. *Pergamon, Oxford*, 1962.

4. Kabaila V., Some properties of the functions of the class H'_p and the interpolation problems. (Russian) *Lit. Math. Notes.* X, **3** (1970), 488–495.

5. Muskhelishvili N. Singular Integral Equations. *Noordhoff, Groningen, Leyden*, 1953.

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