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STATIONARY LIQUID FLOWS BETWEEN TWO NONCOAXIAL PERMEABLE/NONPERMEABLE CYLINDRICAL SURFACES

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Abstract. The paper considers different stationary liquid flows between noncoaxial permeable/nonpermeable cylindrical surfaces which are free from external forces. Using the well-known method [5], the mathematical apparatus for mapping two nonintersecting circumferences onto two concentric circumferences is constructed [5].

Keywords and phrases: Noncoaxial permeable cylinders, stationary flow, conformal mapping.

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The stationary liquid flows between two coaxial cylinders have been investigated by many scientists [1]–[4]. As for stationary liquid flows between two noncoaxial permeable/nonpermeable cylinders, they are not investigated even in a simplest case when one cylinder is in the other noncoaxial one. The aim of the present work is to construct a mathematical apparatus for mapping two nonintersecting circumferences onto two concentric ones [5]. In a general case the analytic functions have to be constructed (Figs. 1,2,3) [5].

Let us first consider the mappings under which the both circumferences transform into themselves. Such transformations are those under which the fixed points are the nodes, while the trajectories are the given circumferences. At this step we can achieve that any of two given circumferences of an orthogonal family pass into one another [5].

In a general case, to transform a pair of circumferences $K_{z,1}$ and $K_{z,2}$ on the plane z into a pair of circumferences $K_{w,1}$ and $K_{w,2}$ on the plane w we can draw through every of these pairs of circumferences an orthogonal circumference in such a way that the latter ones pass into one another, and the points z_1, z_1', z_2, z_2' pass into the points w_1, w_1', w_2, w_2' ([5]). Conversely, if these four points under the linear-fractional transformation pass one into another, then the orthogonal circumferences and the pairs of the given circumferences pass likewise one into another. For this to be the case, the cross-ratios of this set of fours should coincide.

As is known [5], the linear-fractional functions $w = [ax + b]/[cz + d]$, $c \neq 0$, for which the determinant $D = ad - bc$ is other than zero [5], map a sphere (an extended complex plane) conformally and in a one-to-one manner onto itself and preserve circumferences, i.e., transform straight lines and circumferences into straight lines and circumferences. Mutually orthogonal circumferences are transformed again into orthogonal circumferences. Substituting in this equation $w = z = \zeta$ and taking into account that $D = ad - bc = 1$, we obtain $c\zeta^2 - (a-d)\zeta - b = 0$, $\zeta_{1,2} = (a-d) \pm \sqrt{(a+d)^2 - 4})/[2c]$. If $a + d \neq 2$, we obtain two different fixed points ζ_1 and ζ_2, and the equation $w = [ax + b]/[cz + d]$ is reduced to the form $[w - \zeta_1]/[w - \zeta_2] = s[z - \zeta_1]/[z - \zeta_2]$ ([5]). Thus we can find s: $s = [c\zeta_2 + d]/[c\zeta_1 + d] = [c\zeta_2 + d]^2 = [a + d - \sqrt{(a+d)^2 - 4}]^2/4$.

Introducing the notation $W = [w - \zeta_1]/[w - \zeta_2]$, $Z = [z - \zeta_1]/[z - \zeta_2]$, we obtain $W = sZ$.

Let us consider three cases. 1. $s > 0$. The trajectories are the circumferences passing through ζ_1 and ζ_2 (hyperbolic transformations). If $s > 1$, then all points move away from ζ_1 and approach the point $k\zeta_2$; if $s < 0$, everything is vice versa.

2. $|s| = 1$. The trajectories are orthogonal to the circumferences passing through the points ζ_1 and ζ_2. The transformation is called elliptic. There remains the cross-ratio

$$[w_1 - w_3]/[w_1 - w_4] = [w_2 - w_3]/[w_2 - w_4]$$
$$= [z_1 - z_3]/[z_1 - z_4] : [z_2 - z_3]/[z_2 - z_4].$$

The mapping is hyperbolic if $a + d$ is real and $|a + d| > 2$; it is elliptic, if $a + d$ is real and $|a + d| < 2$ ([5]).

Using the above-described construction, we find the angles ζ_1 and ζ_2 corresponding to those circumferences. The transformation $w = s[z-\zeta_1]/[z-\zeta_2]$ transforms the given circumferences on the plane z into the concentric circumferences on the plane w. In the capacity of s we can take any complex number; for example, it can be defined in such a way that one of the concentric circumferences has the given radius and the given point of one of the initial circumferences transfers to the given point of the corresponding circumference. Interchanging the angles ζ_1 and ζ_2, the outer circumference on the plane w becomes inner one, and vice versa.

First, we have to find a cross-ratio. Putting

$$\Delta(K_{z,1}, K_{z,2}) = [(z - z_2)/(z_1' - z_2)] : [(z - z_2')/(z_1' - z_2')],$$

we have

$$\Delta(K_{z,1}, K_{z,2}) = \Delta(K_{w,1}, K_{w,2}).$$

Let $K_{z,1}$ and $K_{z,2}$ be the given circumferences, M_1 and M_2 be the centers, r_1 and r_2 be the radii, d be the distance between M_1 and M_2. We draw through M_1 and M_2 the straight line and denote the points of its intersection with the circumferences by z_1, z_1', z_2, z_2'. Writing out by these points the value $\Delta(K_{z,1}, K_{z,2})$, for the nonintersecting circumferences (Fig. 1) [5] we obtain

$$\Delta(K_{z,1}, K_{z,2}) = [d^2 - (r_1 + r_2)^2]/[d^2 - (r_1 - r_2)^2]. \tag{1}$$

If the circumferences lie one into another, then (Fig. 2)

$$\Delta(K_{z,1}, K_{z,2}) = [(z_1 - r_1)^2 - d^2]/[(z_2 + r_1)^2 - d^2]. \tag{2}$$

If one of the circumferences, say $K_{z,2}$, degenerates into the straight line (Fig. 3), then we simply have

$$\Delta(K_{z,1}, K_{z,2}) = [a - r_1]/[a + r_1], \tag{3}$$

where a is the distance from M_1 to the straight line $K_{z,2}$. Assume that the straight line M_1M_2 is mapped onto the axis so that $K_{z,1}$ is mapped into the unit circumference, and $K_{z,2}$ into the circumference $|w| = R > 1$. Let, moreover, the points z_1, z_1', z_2, z_2' be mapped into the points $w_1 = 1$, $w_1' = -1$, $w_2 = R$, $w_2' = -R$ (Fig. 4). Then

$$\Delta(K_{w,1}, K_{w,2})[(R - 1)/(R + 1)]^2.$$

Since this expression upon the mapping remains unchanged, we have

$$R = [1+\sqrt{\Delta}]/[1-\sqrt{\Delta}].$$

Note that we take the positive value of the root from the corresponding value of Δ: (1), (2) or (3) ([5]).

According to $[w-w_1]/[w-w_2]$:

$$[w_3-w_4]/[w_3-w_2] = [z-z_1]/[z-z_2] : [z_3-z_1]/[z_3-z_2],$$

The unknown function can be written in the form

$$[[w-1]/[w+1]] \cdot [R+1]/[R-1] = [[z-z_1]/[z_1-z_1']] \cdot [z_2-z']/[z_2-z_1].$$

Solving with respect to w, we obtain

$$w = [(\Delta_1+1)z(z_1'\Delta+z_1)]/[(\Delta-1)z-(z_1'-z_1)], \quad \Delta = \frac{z_1-z_2}{z_1'-z_2}\cdot\frac{1}{\sqrt{\Delta}}. \tag{4}$$

For the cases (1), (2) and (3) we obtain

$$\Delta = \sqrt{[(d-z_2)^2-r_1^2]/[(d+r_1)^2-z_1^2]},$$
$$\Delta = \sqrt{[d_2^2-(d+r_1)^2]/[r_2^2-(d-r_1)^2]}, \quad \Delta = \sqrt{[a-r_1]/[a+r_1]},$$

respectively.

Equality (4) allows us to find the nodes of the elliptic net defined by the circumferences $K_{z,1}$ and $K_{z,2}$. Since the points $w=0$ and $w=\infty$ correspond to those nodes [5],

$$\zeta_1 = [z_1'\Delta+z_1]/[A+1], \quad \zeta_2 = [z_1'\Delta-z_1]/[A-1].$$

To solve the problem we have to consider three cases of noncoaxisial cylinders (Figs. 1,2,3). Using the conformal mapping, we reduce the cases given in the Fig. 1,2,3 to the case of stationary liquid motion between two concentric cylinders (see, [1], §15, p. 311-316). Let the liquid occupy the space between circular coaxial cylinders of radii r_1 and r_2 (see, [1], Fig. 114, §15) rotating round the general axis with constant angular velocities ω_1 and ω_2. Define the liquid motion assuming that the motion is stationary and the external forces are absent. The consideration of the viscous liquid motion in the cylindrical system of coordinates under the condition $v_z = v_r = 0$, $v_\vartheta = v(r)$, $P = P(r)$ is facilitated considerably by solving the Novier-Stokes equation which is reduced to the Euler type equation with respect to v

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{v^2}{r}, \quad \frac{d^2v}{dr^1} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = 0,$$

which is solved to the end [1].

The obtained results [1] can be generalized to the cases considered in [1]–[4].

Further we have to establish analytic connection between the figures presented in [5] (Fig. 1–4, p. 85–86) and in [1] (Fig. 114, § 5, p. 344–346).

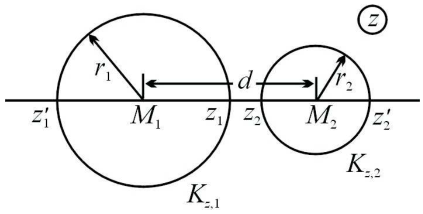

Fig. 1

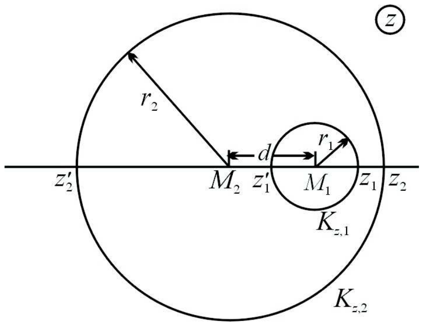

Fig. 2

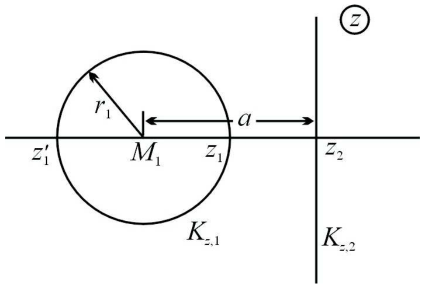

Fig. 3

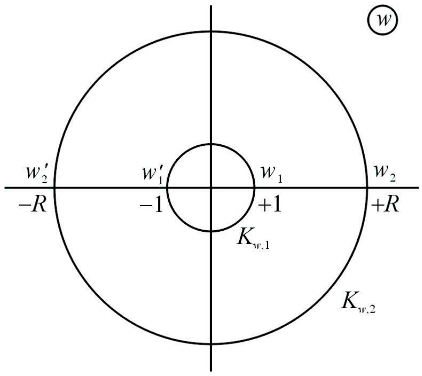

Fig. 4

REFERENCES

1. Kochin N.E., Libel I.A., Rose N.V. Theoretical Hydromechanics. 3rd edition. (Russian) *OGIZ. Gos. Izdar. Techn.-theoretic Liter, Moscow-Leningrad, Part II*, 1948, 344-346.

2. Dolidze D.E. Rotation of a porous cylinder in a liquid. (Russian) *Trudy Tbiliss. Gos. Univ. Univ. Press*, (1963), 39-41.

3. Chandrasekhar S., Hydrodynamic and Hydromagnetic stability. Clarendon, Oxford, 1961.

4. Shapakidze L.V. Influence of rotation on the stability of flow between rotating cylinders. *Proc. Tbiliss. Mat. Inst.*, **96** (1991), 123-138.

5. Kopenfelds V., Stelman F. Practice of conformal mappings, *German transl. Izdat. Inostr. Literat. Moscow*, 1963, 70-87.

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