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NUMBER OF COMPONENTS OF THE ZERO-SET OF QUATERNIONIC POLYNOMIAL

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Abstract. The topological structure of the zero-sets of quaternionic polynomials is discussed. It will be shown that their zero-sets consists of a finite number of points and two-dimensional Euclidean spheres. The effective method of counting the components of both types is also described.

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We deal with polynomials of one variable over the algebra of quaternions [1]

$$H = \{q : q = a+bi+cj+dk, \ \ \forall\, a,b,c,d \in R, \ \ i^2 = j^2 = k^2 = -1, \ \ ij = -ji = k, \dots\}.$$

We consider the so-called standard quaternionic polynomials

$$P(q) = \sum_{m=0}^{n} \alpha_m\, q^m, \quad \text{where} \quad \alpha_i \in H \quad \text{and} \quad \alpha_n \neq 0. \tag{$*$}$$

It is obvious that the problems about roots of that polynomial are reduced to the case of the so-called monic polynomial, i.e., to the case, where $\alpha_n = 1$. Throughout this article, P will be assumed to be a monic polynomial. Properties of its roots will be studied by the topological methods with the use of the theory of topological mapping degree [2]. In the sequel we essentially use the following important fact.

Proposition 1. (cf. [3]) *Every quaternion satisfies a quadratic equation with real coefficients.*

More precisely, it can directly be verified that a quaternion $\alpha = a + bi + cj + dk$ satisfies the quadratic equation with real coefficients:

$$q^2 - 2aq + a^2 + b^2 + c^2 + d^2 = q^2 - 2Re(\alpha)q + Nr(\alpha) = 0.$$

The polynomial

$$P_\xi = q^2 - 2Re(\alpha)q + Nr(\alpha)$$

is called the characteristic polynomial of the quaternion α and is an irreducible quadratic trinomial from the ring of polynomials $\mathbb{R}[x]$. The converse statement is also valid: if $g(q) = q^2 + 2tq + s$ is a quadratic trinomial with a negative discriminant, then any quaternion $\beta = a' + b'i + c'j + d'k$, for which $a' = Re(\beta) = -t$, and $Nr(\beta) = s$, is a root of the polynomial g.

Lemma 1. *For any quaternion polynomial $P(q)$ with $deg(P(q)) \geq 2$ and any $\xi \in H$, there exist polynomials $Q(q)$ and $L(q)$ such that*

$$P(q) = Q(q)P_\xi + L(q),$$

and either $deg(L(q)) \leq 1$, or $L(q) \equiv 0$. Polynomials $Q(q)$ and $L(q)$, satisfying that equation, are defined uniquely.

Since coefficients of each characteristic polynomial are real, there is no need to prove this lemma because its proof is identical to that of the theorem on the divisibility of two polynomials with real coefficients (see [4]). We also need to introduce the notion of conjugate polynomial for (*). Denote by $\overline{P}$ the polynomial of the type

$$\overline{P}(q) = \sum_{m=0}^{n} \overline{\alpha}_m q^m,$$

i.e., the polynomial which is obtained from P by replacing the coefficients α_i, $i = \overline{0, n}$, by their conjugates $\overline{\alpha_i}$.

Determine now an auxiliary polynomial with real coefficients which will allow us to investigate the roots of a given polynomial. Namely, we put $N(P) = P \cdot \overline{P}$, where the polynomial $N(P)$ is obtained by multiplying P by $\overline{P}$ according to the rule that the unknown q commutes with coefficients α_i and $\overline{\alpha_i}$, $i = \overline{0, n}$.

Definition 1. Polynomial $N(P)$ is called the quasi-norm of P.

Lemma 2. ([5]) *The quasi-norm $N(P)$ of an arbitrary canonical polynomial P is a polynomial with real coefficients of degree $2deg(P)$.*

Lemma 3. *Let there be given the polynomial P and the quaternion ξ. Then we have a dichotomy: either P_ξ divides P and then the whole $[\xi]$ consists of roots of P, or there is no more than one root of P in $[\xi]$, where P_ξ is the characteristic polynomial of the quaternion ξ, and $[\xi]$ is the class of similar quaternions ξ.*

Corollary. The set of roots of a polynomial P is infinite if and only if there exists $\xi \in \mathbb{H}$ such that P is divisible by P_ξ.

Lemma 4. ([5]) *If ξ is a root of the polynomial P, then the characteristic polynomial $P(\xi)$ of the quaternion ξ divides $N(P)$.*

Theorem 1. ([6]) *The zero-set of a canonical quaternion polynomial*

$$P(q) = \sum_{m=0}^{n} \alpha_m q^m, \quad \alpha_i \in \mathbb{H}, \quad i = \overline{0, n}, \quad \alpha_n = 1,$$

consists of t isolated points and $s \leq \frac{n-t}{2}$ two-dimensional spheres, i.e. the inequality $t + 2s \leq n$ is valid.

We introduce now the notion of multiplicity of a component of the set of roots. For a component consisting of one point one can apply the standard definition of pre-image multiplicity under smooth mapping [7], [8]. For the sake of simplicity we will speak about pre-images of zero only, i.e., about roots. Then, according to what has been said above, we arrive at the following definition of multiplicity of an isolated quaternion root.

Definition 2. (cf. [9], [10]) Multiplicity of an isolated root of a quaternion polynomial is defined as the dimension of the local algebra of the corresponding polynomial endomorphism of H.

Definition 3. Algebraic multiplicity of an isolated root of the polynomial P is defined as the exponent with which the characteristic polynomial of the given root is involved in the factorization of the quasi-norm $N(P)$. Algebraic multiplicity of a two-dimensional component of the zero-set is defined as the half of the exponent with which the characteristic polynomial of the given component is involved in the factorization of the quasi-norm $N(P)$.

For the sake of convenience, the number obtained in such a way will be called the *algebraic multiplicity* of the component in question. Moreover, it turns out that the geometric multiplicity of an isolated root can be easily expressed by its algebraic multiplicity.

Proposition 2. ([5]) *Geometric multiplicity of an isolated root of a canonical quaternion polynomial is equal to the fourth degree of the algebraic multiplicity of that root.*

Theorem 2. ([5]) *For any canonical quaternion polynomial P, the sum of algebraic multiplicities of all components of its zero-set Z_P is equal to the algebraic degree of P.*

Here we describe the method which allows one to establish the existence of continual components and to estimate their number, not evaluating roots and solutions of one or another system of polynomial equations.

By the coefficient of the given canonical polynomial P we can construct algorithmically a pair of real polynomials of two variables $N(x, y)$, $T(x, y)$ [11], such that if q_0 is an arbitrary root of the equation $P(q) = 0$, then its trace and norm are the real solutions of the system of equations $\{N(x, y) = 0,\ T(x, y) = 0\}$. In addition, it is known that the given system has always a finite set of real solutions.

Thus to establish the existence of spherical components, it is sufficient to find out whether the Niven's system has multiple solutions. To this end, we denote by $J = J(N, T)$ the Jacobian of that pair of polynomials and note that multiple solutions are characterized by the fact that the Jacobians of the system vanish.

Proposition 3. *If the extended Niven's system $\{N = 0,\ T = 0,\ J = 0\}$ has no real solutions, then a set of roots of the given polynomial consists of isolated points.*

Canonical polynomial P is representable in the norm

$$P(q) = Q(q) \cdot P_\xi(q) + F_p(t, n) \cdot q + G_p(t, n),$$

where t and n are the coefficients of the characteristic polynomial P_ξ, i.e., the trace and norm of an arbitrary quaternion from the preassigned class of conjugacy, and F_p and C_p are some quaternion-valued functions, depending on the above-introduced real variables t and n.

Proposition 4. *For a polynomial P, the function F_p and C_p are the polynomials of the variables t and n, and as P varies, they are the polynomial functions of coefficients of the polynomial P.*

Theorem 3. *The number s of roots of the system $\{F_p(t, n) = 0,\ G_p(t, n) = 0\}$, lying in the upper half-plane $\{n > 0\}$, can algorithmically be found by means of a finite number of algebraic and logical operations over the coefficients of the given canonical* *polynomial P, and coincides with a number of spherical components of the set of roots of the polynomial P.*

Corollary. A number s, suggested by the theorem, is equal to zero, if and only if the set of roots of the polynomial P does not contain spherical components.

Since a number s can easily be calculated, we can conclude that a number of spherical components can also be calculated easily in every particular case. It should be emphasized that the number determined in such a way coincides with that of geometrically distinct spherical components of the set of roots.

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