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TWO-DIMENSIONAL MODIFIED BOUNDARY VALUE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURES

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Abstract. In the paper, two-dimensional modified first boundary value problem of statics of elastic mixtures is investigated. We prove that, the problem for multiply-connected finite domain has a unique solution, which is representable by double layer general logarithmic potential with complex densities.

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1^0. The homogeneous equation of statics of the theory of elastic mixture in the complex form is written as [2].

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + \mathcal{K}\frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \quad U = (u_1 + iu_2, u_3 + iu_4)^T, \tag{1}$$

where u_p, $p = \overline{1,4}$, are components of the partial displacement vector,

$$\mathcal{K} = -\frac{1}{2}em^{-1}, \quad e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \quad m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix},$$
$$m_k = e_k + \frac{1}{2}e_{3+k}, \quad k = 1, 2, 3,$$

the e_q, $q = \overline{1,6}$ are expressed in terms of the elastic constants [2].

In [1] M. Basheleishvili obtained the representations

$$U = m\varphi(z) + \frac{1}{2}ez\overline{\varphi'(z)} + \overline{\psi(z)}, \quad TU = \frac{\partial}{\partial s(x)}[-2\varphi(z) + 2\mu U(x)],$$

where $TU = ((Tu)_2 - i(Tu)_1), (Tu)_4 - i(Tu)_3)^T$, $(Tu)_p$, $p = \overline{1,4}$, are components of the stress vector, $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions, $\mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}$, μ_k, $k = 1, 2, 3$, are elastic constants, $\frac{\partial}{\partial s(x)} = n_1\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_1}$, $n = (n_1, n_2)$ is an arbitrary unit vector.

To investigate modified problem, the use will be made of the following vectors [2] $V = i[-m\varphi(z) + \frac{1}{2}ez\overline{\varphi'(z)} + \overline{\psi(z)}]$,

$$NU = TU - (2\mu - m^{-1})\frac{\partial U}{\partial s(x)}, \quad NV = i\frac{\partial}{\partial s(x)}[2\varphi(z) - im^{-1}V(x)],$$

where V is a vector associated to U, NU is the pseudo-stress vector [2].

It is not difficult to prove that V satisfies (1), moreover

$$NU = -im^{-1}\frac{\partial V}{\partial s(x)}, \quad NV = im^{-1}\frac{\partial U}{\partial s(x)}. \tag{2}$$

Let D^+ is a finite multiply connected domain bounded by the closed contours $\Gamma_p \in C^{2,\beta}$, $0 < \beta \le 1$, $p = \overline{0,n}$, where Γ_0 contains all the others. In this case, the boundary of D^+ is $\Gamma = \bigcup\limits_{p=0}^{n}\Gamma_p$; note that contours Γ_p $(p \le n)$ are oriented clockwise, while Γ_0 is oriented counterclockwise. Let D_p^-, $p = \overline{1,n}$, be a finite two-dimensional domain bounded by the contour Γ_p, $p = \overline{1,n}$. By D_*^- we denote unbounded domain which boundary is Γ_0. Let D_0^- be a infinite domain bounded by the contour Γ_0.

In the sequel we assumed that $u_j \in C^2(D^+) \cap C^{1,\alpha}(\overline{D}^+)$, $0 < \alpha < \beta \le 1$, $j = \overline{1,4}$.

If U is a regular solution to (1), then we have the following Green formula [2]

$$\int_{D^+} N(u,u)dy = \operatorname{Im}\int_\Gamma U\overline{NU}ds = \operatorname{Im}\int_\Gamma V\overline{NV}ds, \tag{3}$$

where $N(u,u)$ is positive quadratic form [3]. We need the following

Lemma1. *Solution of the equation $N(u,u) = 0$ is an arbitrary constant vector.*

2^0. Consider the following boundary value problems. Find a regular solution to the equation (1) in D^+, satisfying one of the following conditions

Problem A:

$$U(t) = f(t), \quad \text{on} \quad \Gamma,$$

Problem B:

$$U(t) = f(t) + \alpha(t), \quad \text{on} \quad \Gamma,$$

where $f = (f_1, f_2)^T \in C^{1,\alpha}(\Gamma)$, $0 < \alpha < \beta \le 1$, is given vector-function,

$$\alpha(t) = \alpha_p, \quad \text{on} \quad \Gamma_p, \quad p = \overline{0,n}$$

α_p is a two-dimensional constant vector, $p = \overline{0,n}$, which will be defined later in solving the problem, if we fix one of them. Bellow we will assume that $\alpha_0 = 0$.

As known problem A is the first plane boundary value problem of statics of elastic mixtures. Problem B we call the modified boundary value problem of A. Similar problem - "Modified problem Dirichlet" have be investigate N. Muskhelishvili [4] §60–§65.

The following theorem is valued.

Theorem 1. *The homogeneous problems A^0 and B^0 have only the trivial solution.*

We look for solution to the problem B in the form of a double layer general logarithmic potential with complex densities

$$U(x,g) = \frac{1}{\pi}\int_\Gamma \frac{\partial \ln|z-\varsigma|}{\partial n(y)} g(y)dyS + \frac{\mathcal{K}}{2\pi i}\int_\Gamma \frac{\partial}{\partial s(y)}\frac{z-\varsigma}{\overline{z}-\overline{\varsigma}}\overline{g(y)}dyS, \tag{4}$$

where $z = x_1 + ix_2$, $\varsigma = y_1 + iy_2$, $g = (g_1, g_2)^T \in C^{1,\alpha}(\Gamma)$, $0 < \alpha < \beta \le 1$ is unknown complex vector.

Then we arrive to the following Fredholm integral equation

$$g(t)+\frac{1}{\pi}\int_{\Gamma}\frac{\partial\ln|t-\varsigma|}{\partial n(y)}g(y)\,dyS+\frac{\mathcal{K}}{2\pi i}\int_{\Gamma}\frac{\partial}{\partial s(y)}\frac{t-\varsigma}{\overline{t}-\overline{\varsigma}}\overline{g(y)}dyS=f(t)+\alpha(t), \tag{5}$$

where $t\in\Gamma$, $f\in C^{1,\alpha}(\Gamma)$, $0<\alpha<\beta\leq 1$, $\alpha(t)=\alpha_p=\text{const}$ on Γ_p, $p=\overline{0,n}$, $\alpha_0=0$.

To investigate (5), its advisable to consider, instead of (5), the equation

$$\begin{aligned} g(t)+\frac{1}{\pi}\int_{\Gamma}\frac{\partial\ln|t-\varsigma|}{\partial n(y)}\,g(y)\,dyS+\frac{\mathcal{K}}{2\pi i}\int_{\Gamma}\frac{\partial}{\partial s(y)}\frac{t-\varsigma}{\overline{t}-\overline{\varsigma}}\,\overline{g(y)}\,dyS \\ -\int_{\Gamma}H(t,y)g(y)d_yS=f(t), \end{aligned} \tag{6}$$

where

$$H(t,y)=\begin{cases} h_p(y), & \text{when}\quad (t,y)\in\Gamma_p,\quad p=\overline{1,n},\\ 0, & \text{in other case.}\end{cases}$$

Here $h_p(y)$, $y\in\Gamma_p$ is a real continuous function and satisfying the condition

$$\int_{\Gamma_p}h_pds\neq 0,\quad p=\overline{1,n}.$$

At every on the contour Γ_p

$$\int_{\Gamma}H(t,y)g(y)dyS=dp,\quad\text{when}\quad t\in\Gamma_p,\quad p=\overline{0,n},\quad d_0=0, \tag{7}$$

where d_p is two-dimensional constant vector, namely $d_p=\int\limits_{\Gamma_p}h_pgds$.

Let us prove that equation (6) is always solvable. Assume the contrary and denote any solution of the homogeneous equation corresponding to (6) by g_0 i.e.

$$\begin{aligned} g_0(t)+\frac{1}{\pi}\int_{\Gamma}\frac{\partial\ln|t-\varsigma|}{\partial n(y)}\,g_0(y)dyS+\frac{\mathcal{K}}{2\pi i}\int_{\Gamma}\frac{\partial}{\partial s(y)}\frac{t-\varsigma}{\overline{t}-\overline{\varsigma}}\,\overline{g_0(y)}dyS \\ -\int_{\Gamma}H(t,y)g_0(y)d_yS=0. \end{aligned} \tag{8}$$

Then for the double layer potential $U(x,g_0)=U_0(x)$ we have $U_0(x)=d_p$, on Γ_p, $p=\overline{0,n}$, $d_0=0$.

Thus, in view of (2) we obtain $NV_0(t)=0$, on Γ_p, $p=\overline{0,n}$, where $V_0(x)$ is associated to $U_0(x)$ and

$$V_0(x)=-\frac{1}{\pi}\int_{\Gamma}\frac{\partial\ln|z-\varsigma|}{\partial s(y)}\,g_0(y)d_ys+\frac{\mathcal{K}}{2\pi}\int_{\Gamma}\frac{\partial}{\partial s(y)}\frac{z-\varsigma}{\overline{z}-\overline{\varsigma}}\,g_0(y)d_yS.$$

Using (3) we obtain $N(u_0,u_0)=0$, $x\in D^+$, whence by Lemma 1 and $U_0=0$ on Γ_0, we get $U_0(x)=0$, $x\in D^+$. Form the results we can write (see (2)) $V_0(x)=l_0=\text{const}$, $x\in D^+$. As for as $V_0(x)$ passes continuously through the boundary Γ the equality $V_0(x)=l$, $x\in D^+$ allows one to write $V_0^{\pm}(t)=l_0$, $t=\Gamma_p$, $p=\overline{0,n}$.

Since $V_0^-(t) = l_0$, $t \in \Gamma_0$ and $V_0(\infty) = 0$, therefore by the uniqueness theorem for the first external problem of statics of elastic mixtures [2] we have $V_0(x) = 0$, $x \in D^+$. Hence $V_0(x) = 0$, $x \in D^+$ and $x \in D_p^-$, $p = \overline{1,n}$.

Applying now Green's formula, we obtain $U_0(x) = l_p = \text{const}$, $x \in D_p^-$, $p = \overline{0,n}$. But since $U_0(\infty) = 0$, we get

$$U_0(x) = 0, \quad x \in D_0^- \quad \text{and} \quad U_0(x) = l_p, \quad x \in D_p^-, \quad p = \overline{1,n}. \tag{9}$$

Taking into account that $U_0^+(t) - U_0^-(t) = 2g_0$, $t \in \Gamma_p$, $p = \overline{0,n}$, we obtain

$$g_0(t) = 0, \quad t \in \Gamma_0 \quad \text{and} \quad g_0(t) = l_p, \quad p = \overline{1,n}. \tag{10}$$

The substitution of (10) into (12) we get $l_p \int\limits_{\Gamma_p} h_p ds = 0$, $p = \overline{1,n}$, whence by (10) have $l_p = 0$, $p = \overline{1,n}$, consequently $g_0 = 0$, on Γ_p, $p = \overline{0,n}$.

Thus the equation (6) is uniquely solvable and $g \in C^{1,\alpha}(\Gamma)$ $0 < \alpha < \theta \leq 1$.

Now note that solution of the equation (6) gives the solution of the initial equation (5) because by virtue of (7) we have

$$g(t) + \frac{1}{\pi} \int_\Gamma \frac{\partial}{\partial n(y)} \ln|t - \varsigma| g(y) dyS + \frac{\mathcal{K}}{2\pi i} \int_\Gamma \frac{\partial}{\partial s(y)} \frac{t - \varsigma}{\overline{t} - \overline{\varsigma}} \overline{g(y)} dyS$$
$$= f(t) + d_p, \quad \text{on} \quad \Gamma_p,$$

Moreover in equation (4) partner constant vectors α_p, $p = \overline{1,n}$, are defined by

$$\alpha_p = d_p = \int_{\Gamma_p} h_p g ds, \quad p = \overline{1,n}$$

Thus, problem B in the domain D^+ has unique regular solution and it is representable by (4).

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