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ON A VERSION OF NON-LOCAL BITSADZE-SAMARSKY PROBLEM FOR LINEAR MIXED TYPE EQUATION

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Abstract. For second order mixed elliptic-hyperbolic type equation with strong parabolic degeneracy the non-local boundary value problem is investigated.

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In the A. Bitsadze and A. Samarsky's joint paper [1] the non-classical non-local boundary value problem in rectangular domain $D_1\{-\ell < x < \ell,\ 0 < y < 1\}$ for the Laplace equation is posed and investigated. According to this problem values of sought solution in certain sets of internal and boundary points are connected.

In present paper the cognate non-local boundary value problem for second order mixed type equation

$$u_{xx} + y\, u_{yy} + b\, u_y = 0, \quad b > 1, \quad b \neq [b]. \tag{1}$$

In rectangular domain $D\{0 < x < \ell,\ -q < y < p\}$ with $\ell, p, q > 0$ given numbers is considered. In sub-domains $D^- = D \cap (y < 0)$ and $D^+ = D \cap (y > 0)$ (1) belongs to the classes of hyperbolic and elliptic equations correspondingly and is parabolically degenerated on the interval $0 \le x \le \ell$, $y = 0$. This interval of type-degeneracy is the envelope of the both of families of characteristics and simultaneously itself is the singular characteristic of given equation. In such case (1) have the strong characteristic parabolic degeneracy and both of solution and it's first order derivatives a priori can be unbounded in the neighbourhood of the interval of degeneracy. The order of their growth is defined by the lowest term of (1) and in general the solution of the first boundary problem does not exist. For such cases A. Bitsadze did suggest some weighted conditions for solution and its first order derivative in respect to the argument y along the curve of degeneracy. Taking into account this factor below the following Bitsadze-Samarsky type non-local problem is considered:

It is to find twice continuously differentiable in sub-domains D^+, D^- solution $u(x,y)$ of (1) subordinate to assembling relations

$$\lim_{y\to 0_+} y^{b-1}u^+(x,y) = \lim_{y\to 0_-}(-y)^{b-1}u^-(x,y), \ \ \lim_{y\to 0_+} y^{b}u_y^+(x,y) = \lim_{y\to 0_-}(-y)^{b}u_y^-(x,y), \tag{2}$$

and satisfying homogeneous boundary

$$u(x,y)\big|_{x=0} = u(x,y)\big|_{x=\ell} = 0 \tag{3}$$

and non-local conditions

$$u^+(x,p) + u^-(x,-\rho) = \varphi(x), \quad 0 \le x \le \ell, \quad \varphi(0) = \varphi(\ell) = 0, \tag{4}$$

$$u^+(x,r)+u^-(x,-q)=f(x),\quad 0\le x\le \ell,\quad f(0)=f(\ell)=0. \tag{5}$$

According to the boundary conditions (3) the following equalities $\varphi(0)=\varphi(\ell)=0$ and $f(0)=f(\ell)=0$ should be fulfilled. We demand from the functions $\Phi(x)$ and $F(x)$ which are obtained from $\varphi(x)$ and $f(x)$ by the oddness law on the segment $[-\ell,0]$ and then by their prolongation on the whole axis with period 2ℓ, to be two times continuously differentiable. We also demand, the following $\varphi^k(0)=\varphi^k(\ell)$, $f^k(0)=f^k(\ell)$ $(k=0,2)$. Such periodical functions when they expansion into a Fourier series, their coefficients tend to zero by order $O(n^{-2})$ [3].

Taking into consideration the conditions (3) for the equation (1), the solution Dirichlet problem is represented by the Fourier-Bessel series [6–7]:

$$u^+(x,y)=\sum_{n=0}^{\infty}(2\sqrt{y})^{1-b}\Big[a_nI_{1-b}\Big(\frac{2\pi n}{\ell}\sqrt{y}\Big)+b_nI_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{y}\Big)\Big]\sin\frac{\pi n}{\ell}x, \tag{6}$$

$$u^-(x,y)=\sum_{n=0}^{\infty}(2\sqrt{-y})^{1-b}\Big[k_nJ_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{-y}\Big)+d_nJ_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{-y}\Big)\Big]\sin\frac{\pi n}{\ell}x, \tag{7}$$

where $J_{1-b},J_{b-1},I_{1-b},I_{b-1}$ are Bassel functions and a_n,b_n,d_n,k_n are the constants of integrating. If we satisfy the conditions (2), (4), (5) of the problem by the series (6) and (7), we shall obtain the linear inhomogeneous algebraic system relative to these constants and for natural values n:

$$\begin{aligned}
&\frac{\big(\frac{2\pi n}{\ell}\big)^{1-b}}{\Gamma(2-b)}a_n-\frac{\big(\frac{2\pi n}{\ell}\big)^{1-b}}{\Gamma(2-b)}k_n=0,\quad \frac{2^{1-b}\big(\frac{2\pi n}{\ell}\big)^{-b}}{\Gamma(1-b)}a_n+\frac{2^{1-b}\big(\frac{2\pi n}{\ell}\big)^{-b}}{\Gamma(1-b)}k_n=0,\\
&(2\sqrt{p})^{1-b}\Big[a_nI_{1-b}\Big(\frac{2\pi n}{\ell}\sqrt{p}\Big)+b_nI_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{p}\Big)\Big]\\
&\quad+(2\sqrt{\rho})^{1-b}\Big[k_nJ_{1-b}\Big(\frac{2\pi n}{\ell}\sqrt{\rho}\Big)+d_nJ_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{\rho}\Big)\Big]=\varphi_n,\\
&(2\sqrt{r})^{1-b}\Big[a_nI_{1-b}\Big(\frac{2\pi n}{\ell}\sqrt{r}\Big)+b_nI_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{r}\Big)\Big]\\
&\quad+(2\sqrt{q})^{1-b}\Big[k_nJ_{1-b}\Big(\frac{2\pi n}{\ell}\sqrt{q}\Big)+d_nJ_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{q}\Big)\Big]=f_n,
\end{aligned}\tag{8}$$

where φ_n, f_n are the Fourier coefficients of the functions Φ, F and Γ is the Euler function. From the system (8) it follows $a_n=k_n=0$, while for determining of constants b_n, d_n the following conditions are to be fulfilled:

$$\begin{aligned}
\Delta&=(4\sqrt{pq})^{1-b}I_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{p}\Big)J_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{q}\Big)\\
&\quad-(4\sqrt{r\rho})^{1-b}I_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{r}\Big)J_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{\rho}\Big)\neq 0,\\
b_n&=\frac{\Delta_b}{\Delta}=\frac{(2\sqrt{q})^{1-b}J_{b-1}\big(\frac{2\pi n}{\ell}\sqrt{q}\big)\cdot\varphi_n-(2\sqrt{\rho})^{1-b}J_{b-1}\big(\frac{2\pi n}{\ell}\sqrt{\rho}\big)\cdot f_n}{\Delta},\\
d_n&=\frac{\Delta_d}{\Delta}=\frac{(2\sqrt{p})^{1-b}I_{b-1}\big(\frac{2\pi n}{\ell}\sqrt{p}\big)\cdot f_n-(2\sqrt{r})^{1-b}I_{b-1}\big(\frac{2\pi n}{\ell}\sqrt{r}\big)\cdot\varphi_n}{\Delta}.
\end{aligned}\tag{9}$$

To the functions (6) and (7) we can give the following form:

$$u^+(x,y) = (2\sqrt{y})^{1-b} \sum_{n=0}^{\infty} b_n I_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{y}\Big) \sin\frac{\pi n}{\ell}x = (2\sqrt{y})^{1-b} \sum_{n=0}^{\infty} \frac{\Delta_b}{\Delta_n}\alpha_1 \sin\frac{\pi n}{\ell}x, \tag{10}$$

$$\begin{aligned} u^-(x,y) &= (2\sqrt{-y})^{1-b} \sum_{n=0}^{\infty} d_n J_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{-y}\Big) \sin\frac{\pi n}{\ell}x \\ &= (2\sqrt{-y})^{1-b} \sum_{n=0}^{\infty} \frac{\Delta_d}{\Delta_n} J_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{-y}\Big) \sin\frac{\pi n}{\ell}x, \end{aligned} \tag{11}$$

where

$$\Delta_n = (4\sqrt{pq})^{1-b} J_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{q}\Big) - (4\sqrt{r\rho})^{1-b}\cdot \alpha_2 J_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{\rho}\Big),$$

$$\alpha_1 = \frac{I_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{y}\Big)}{I_{b-1}\big(\frac{2\pi n}{\ell}\sqrt{p}\big)}, \quad \alpha_2 = \frac{I_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{r}\Big)}{I_{b-1}\big(\frac{2\pi n}{\ell}\sqrt{p}\big)}.$$

According to the properties of Bessel functions [4] $0 < \alpha_m < 1$ $(m = 1, 2)$.

In order to show the convergence of the functional series (10) we consider the expression:

$$\begin{aligned} |\Delta_n| &= \Bigg| (4\sqrt{pq})^{1-b} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(b-1+k+1)} \Big(\frac{\pi n}{\ell}\sqrt{q}\Big)^{b-1+2k} \\ &\quad - (4\sqrt{r\rho})^{1-b}\alpha_2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(b-1+k+1)} \Big(\frac{\pi n}{\ell}\sqrt{\rho}\Big)^{b-1+2k} \Bigg| \\ &= \Bigg| \Big(\frac{\pi n}{\ell}\Big)^{b-1} \Bigg\{ (4\sqrt{pq})^{1-b}(\sqrt{q})^{b-1} \Bigg[\frac{1}{\Gamma(b)} - \frac{\big(\frac{\pi n}{\ell}\sqrt{q}\big)^2}{1!\,\Gamma(b+1)} + \dots \Bigg] \\ &\quad - \alpha_2 (4\sqrt{r\rho})^{1-b}(\sqrt{\rho})^{b-1} \Bigg[\frac{1}{\Gamma(b)} - \frac{\big(\frac{\pi n}{\ell}\sqrt{\rho}\big)^2}{1!\,\Gamma(b+1)} + \dots \Bigg] \Bigg\} \Bigg|. \end{aligned}$$

In the square brackets we have the series of Leibniz type [5]. Let us denote the first remainders of these series by $R_1(q)$ and $R_2(\rho)$ the following inequality is known

$$|R_1(q)| < \Bigg| \frac{\big(\frac{\pi n}{\ell}\sqrt{q}\big)^2}{\Gamma(b+1)} \Bigg|, \quad |R_2(\rho)| < \Bigg| \frac{\big(\frac{\pi n}{\ell}\sqrt{\rho}\big)^2}{\Gamma(b+1)} \Bigg|.$$

So we obtain

$$\begin{aligned} |\Delta_n| &\geq \Bigg| \Big(\frac{\pi n}{\ell}\Big)^{b-1} \Bigg\{ (4\sqrt{p})^{1-b} \Bigg[\frac{1}{\Gamma(b)} - \frac{\big(\frac{\pi n}{\ell}\sqrt{q}\big)^2}{1!\,\Gamma(b+1)} \Bigg] - \alpha_2 (4\sqrt{r})^{1-b} \Bigg[\frac{1}{\Gamma(b)} - \frac{\big(\frac{\pi n}{\ell}\sqrt{\rho}\big)^2}{1!\,\Gamma(b+1)} \Bigg] \Bigg\} \\ &\geq \frac{4^{1-b}(\pi n)^{b+1}}{b\,\Gamma(b)\,\ell^{b+1}} \Bigg| \Big| q\sqrt{p}^{1-b} - \alpha_2 \rho \sqrt{r}^{1-b} \Big| - \frac{1}{n^2} \Big| \frac{b\ell^2(\sqrt{p}^{1-b} - \alpha_2\sqrt{r}^{1-b})}{\pi^2} \Big| \Bigg| \end{aligned}$$

beginning at each $n > N$ with some certain number $N > 0$ the inequality

$$\left|q\sqrt{p}^{1-b} - \alpha_2\rho\sqrt{r}^{1-b}\right| \geq \frac{1}{n^2}\left|\frac{b\ell^2(\sqrt{p}^{1-b} - \alpha_2\sqrt{r}^{1-b})}{\pi^2}\right|$$

have to satisfied. Therefore we obtain

$$\left|q\sqrt{p}^{1-b} - \alpha_2\rho\sqrt{r}^{1-b}\right| \geq |\delta_n| \geq \frac{1}{n^2}\left|(q\sqrt{p}^{1-b} - \alpha_2\rho\sqrt{r}^{1-b}) - \frac{b\ell^2(\sqrt{p}^{1-b} - \alpha_2\sqrt{r}^{1-b})}{\pi^2}\right|, \quad n > N,$$

where

$$\delta_n = \left(q\sqrt{p}^{1-b} - \alpha_2\rho\sqrt{r}^{1-b}\right) - \frac{1}{n^2}\frac{b\ell^2(\sqrt{p}^{1-b} - \alpha_2\sqrt{r}^{1-b})}{\pi^2} > 0, \quad n > N,$$

δ_n is increasing convergent sequence. Let us denote $\inf \delta_n = \delta$.

For the series (10) the following equality is fulfilled:

$$\begin{aligned} u^+(x,y) &= (2\sqrt{y})^{1-b}\sum_{n=0}^{\infty}\frac{\Delta_b}{\Delta}I_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{y}\Big)\cdot\sin\frac{\pi n}{\ell}x \\ &\leq \frac{4^{b-1}Mb\Gamma(b)\ell^{b+1}}{\delta}\sum_{n=0}^{\infty}\frac{|\varphi_n - f_n|}{n^{b+1}}, \end{aligned} \tag{12}$$

where

$$M = \sup\left\{(2\sqrt{q})^{1-b}J_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{q}\Big), (2\sqrt{\rho})^{1-b}J_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{\rho}\Big)\right\}.$$

From the convergence of the majorant series (12) it follows that the series (10) are uniformly and absolutely convergent and the function $u^+(x,y)$ is continuous.

For the derivative of the series (10) we have the following inequality

$$\begin{aligned} u_y^+(x,y) &= \sum_{n=0}^{\infty}\frac{\Delta_b}{\Delta_n}\frac{\sqrt{y}^{-b}I_b\big(\frac{2\pi n}{\ell}\sqrt{y}\big)}{(2\sqrt{p})^{1-b}I_{b-1}\big(\frac{2\pi n}{\ell}\sqrt{p}\big)}\cdot\frac{\pi n}{\ell}\cdot\sin\frac{\pi n}{\ell}x \\ &\leq \frac{4^{b-1}Mb\Gamma(b)\ell^{b+1}}{\delta}\sum_{n=0}^{\infty}\frac{\varphi_n - f_n}{n^{b+1}}\cdot\frac{\pi n}{\ell}. \end{aligned}$$

From the last inequality it follows, that the series $u_y^+(x,y)$ are uniformly and absolutely convergent and the function $u_y^+(x,y)$ is continuous. Analogously, we prove that the series $u_{yy}^+(x,y)$ are uniformly and absolutely convergent due to the smoothness of the functions $\Phi(x)$ and $F(x)$. Also, we can show, that series (10) differentiated with respect to variable x, is also convergent.

Let us consider series (11) and show its convergence.

$$u^-(x,y) = \big(2\sqrt{-y}\big)^{1-b}\sum_{n=0}^{\infty}\frac{I_{b-1}\big(\frac{2\pi n}{\ell}\sqrt{p}\big)\Big[(2\sqrt{p})^{1-b}f_n - (2\sqrt{r})^{1-b}\cdot\alpha_2\cdot\varphi_n\Big]}{\Delta}$$

$$\cdot J_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{-y}\Big)\cdot\sin\frac{\pi n}{\ell}x \le \frac{L}{\delta}\sum_{n=0}^{\infty}\Big|(2\sqrt{p})^{1-b}f_n-(2\sqrt{r})^{1-b}\cdot\alpha_2\cdot\varphi_n\Big|,$$

where

$$L=\sup\Big|(2\sqrt{-y})^{1-b}J_{b-1}\Big(\frac{2\pi n}{\ell}\sqrt{-y}\Big)\Big|.$$

From the convergence of the last numerical series it follows that the series (11) are uniformly and absolutely convergent and the function $u^-(x,y)$ is continuous. Analogously it can be proved that first and second derivative of the series (11) with respect to variables x and y are uniformly and absolutely convergent. Thus the following theorem is true.

Theorem. *If the condition* (9) *is fulfilled, then the non-local problem* (1–5) *has the unique solution in the class of the functions, which are representable in the form of the series* (10) *and* (11).

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