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ON AN EXAMPLE FROM THE SPECTRAL REPRESENTATION THEORY OF THE LINEAR MULTIGROUP TRANSPORT PROBLEM

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Abstract. K. Case (1960) introduce a method for solving the neutron transport equation with isotropic scattering. Kanal and Davies (1981) have applied Case's method to the transformation of the original equation of the linear transport theory by expanding the scattering function for the problem to be solved as a spectral integral over the complete set Case's eigenfunctions for a previosly solved transport problem. It was generalized by us these results to the problems of the multigroup transport theory. In this paper the spectral representation of the linear multigroup transport problem is applied to the additional example. We obtain the dispersion relations and eigenfunctions for $(N+1)$-th order scattering by using the eigenfunctions for N-th order scattering as the basis.

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In the paper [1] is developed the mathematical reformulation of singular approach to the solution of the one-dimensional equation of multigroup transport theory. A number of simple examples were presented in which the spectral formulation leads to the standard results of singular approach. In this paper we demonstrate that the eigefunctions for N-th order scattering can be used as a basis set for obtaining the dispersion relation, and eigenfunctions for $(N+1)$-th order scattering.

The phase function for a previously solved transport problem is

$$f_0(\mu \to \mu') = \sum_{s=0}^{N} (2s+1) P_s(\mu) f_s P_s(\mu'), \tag{1}$$

with corresponding characteristic matrix equation

$$(\nu I - \mu \ell)\phi_\nu(\mu) = \frac{c\nu}{2} \int_{-1}^{+1} f_0(\mu' \to \mu)\phi_\nu(\mu') d\mu',$$

and known eigenfunctions $\phi_\nu(\mu)$, eigenvalue spectrum $S_0[\nu]$ and spectral density $d\rho(\nu)$ (see [1]). The phase function for the problem to be solved $(N+1)$-th order scattering is

$$f(\mu \to \mu') = \sum_{s=0}^{N+1} (2s+1) P_s(\mu) f_s P_s(\mu'), \tag{2}$$

with corresponding characteristic matrix equation

$$(\omega I - \mu \ell)\psi_\omega(\mu) = \frac{c\omega}{2} \int_{-1}^{+1} f(\mu' \to \mu)\psi_\omega(\mu') d\mu', \tag{3}$$

and assumed unknown eigenfunctions $\psi_\omega(\mu)$, eigenvalue spectrum $S_{N+1}[\omega]$. Here $\ell = diag\{l_1, ..., l_{i_0}\}, \quad l_i > 0$, moreover without loss of generality we can take $\max_i l_i = 1, \quad P_s(\mu) = diag\{p_s(\mu), ..., p_s(\mu)\}$, $p_s(\mu)$ is the Legendre polynomial of order s, and f_s is $i_0 \times i_0$ matrix, $s = 1, ..., s_0$. For the sake of simplicity, we have chosen f_s are symmetric matrix.

Our basic integral equation (see [1]) is

$$(\omega - \nu)K(\nu, \omega) = \frac{\omega\nu c}{2} \int_{S_0[\nu']} (A(\nu, \nu') - A_0(\nu, \nu'))d\rho(\nu')K(\nu', \omega), \tag{4}$$

where

$$A(\nu, \nu') - A_0(\nu, \nu')$$
$$= \int_{-1}^{+1} d\mu \int_{-1}^{+1} d\mu' \phi_\nu(\mu)(f(\mu' \to \mu) - f_0(\mu' \to \mu))\phi_{\nu'}(\mu').$$

Its solution is equivalent to the solution of equation (3). However, where equation (3) is an integral equation involving an integration over the μ, equation (4) involves the spectral integral over the known eigenvalues of a complete set of solutions to an equation of transport.

Substituting the expressions (1) and (2) into equation (3), and applying some algebraic transformations we find that

$$(\omega - \nu)K(\nu, \omega) = (2N + 3)\frac{c\omega\nu}{2} h_{N+1}^T(\nu) f_{N+1} h_{N+1}(\omega), \tag{5}$$

where superscript T means transpose and

$$h_{N+1}(\nu) = \int_{-1}^{+1} P_{N+1}(\mu)\varphi_\nu(\mu)d\mu.$$

The continuum of $S_0[\nu]$ is known to be given by $-1 \le \nu \le 1$. From equation(4) and (5) we obtain the dispersion relation giving the discrete values of ω, which lie outside of the continuum of $S_0[\nu]$,

$$det\big(I - \frac{c\omega}{2}(2N + 3)\big(L(\omega, S_N)h_{N+1}^T(\omega) + \frac{h_{N+1}(\omega)}{\omega}\big) f_{N+1} g_{N+1}^0(\omega)\big) = 0,$$

where

$$L(\omega, S_N) = \int_{S_N[\nu]} \frac{\nu}{\omega - \nu} h_0(\nu)d\rho(\nu)h_0^T(\nu),$$

$g_{N+1}^0(\omega)$ are defined by $g_0^0(\omega) = I$ and the recursion relation

$$(s + 1)g_{s+1}^0(\omega) + sg_s^0(\omega) = (2s + 1)(I - cf_s)\omega g_s^0(\omega), \qquad s \ge 0,$$

and

$$h_s(\nu) = \int_{-1}^{+1} P_s(\mu)\varphi_\nu(\mu)d\mu.$$

The eigenfunctions $\psi_\omega(\mu)$ for the unsolved problem follow from

$$\psi_\omega(\mu) = \int_{S_0[\nu]} \phi_\nu(\mu) d\rho(\nu) K(\nu,\omega). \tag{6}$$

To obtain $\psi_{\omega_j}(\mu)$ for discrete ω_j, substitute the expression for $K(\nu,\omega_j)$ given by equation (5) into equation (6). We obtain

$$\psi_{\omega_j}(\mu) = (2N+3)\frac{c\omega_j}{2}\int_{S_N[\nu]} \phi_\nu(\mu) d\rho(\nu) \frac{\nu}{\omega-\nu} h^T_{N+1}(\nu) f_{N+1} h_{N+1}(\omega_j).$$

For all ω in the continuum of $S_[\nu], \quad -1 \le \nu \le 1$, we have

$$K(\nu,\omega)M^T_N(\nu,\omega) = -(2N+3)\omega h_{N+1}(\nu) f_{N+1} h_{N+1}(\omega)\varphi^T_\nu(\omega)$$

$$+N_N(\omega)\Lambda(\nu,\omega)M^T_{N+1}(\omega,\omega),$$

where

$$\Lambda(\omega,\mu) = diag\left\{\delta(\omega-\mu l_1), ..., \delta(\omega-\mu l_{i_0})\right\},$$

δ is the Dirac function, $N_N(\omega)$ is the normalization coefficient and

$$M_N(\mu,\omega) = \sum_{s=0}^{N}(2s+1)P_s(\mu) f_s h_s(\omega).$$

In finally we find for continuum ω the eigenfunctions $\psi_\omega(\mu)$ for the unsolved problem

$$\psi_\omega(\mu) = -(2N+3)\omega \int_{S_N[\nu]} \phi_\nu(\mu) d\rho(\nu) h_{N+1}(\nu) f_{N+1} h_{N+1}(\omega)\varphi^T_\nu(\omega) M^{-1}_N(\omega,\nu)$$

$$+\phi_\omega(\mu) M_{N+1}(\omega,\omega) M^{-1}_N(\omega,\omega).$$

REFERENCES

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