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THE EFFECT OF THE TEMPERATURE GRADIENT ON THE STABILITY OF FLOW BETWEEN TWO PERMEABLE CYLINDERS

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Abstract. The linear stability of a viscous fluid flow between two rotating permeable cylinders in the presence of a radial temperature gradient is investigated. The results of numerical calculations for certain parameter values of the problem allow one to study the possible existence of neutral axially symmetric stationary or oscillatory modes.

Keywords and phrases: Stability, temperature gradient, permeable cylinders.

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1. Let a viscous heat-conducting liquid fill the cavity between two rotating permeable cylinders heated to different temperatures. We denote radii, angular velocities and temperatures of the internal and outer cylinders by R_1, Ω_1, Θ_1 and R_2, Ω_2, Θ_2, respectively. Assume that the external mass forces are absent, the velocity flow through the cross-section of the cylinder cavity is equal to zero and the fluid inflow through one cylinders is equal to the fluid outflow through the other.

We use the system of Navier-Stokes, continuity and heat-conductivity equations in terms of the cylindrical coordinates r, φ, z with the axis z coinciding with the axis of the cylinders[1],

$$\begin{gathered}\frac{d\vec{v}'}{dt} = -\frac{1}{\rho}\nabla \operatorname{div} \Pi' + (\nabla \vec{v}' - \nu \operatorname{rot}\operatorname{rot} \vec{v}'), \quad \frac{\partial T'}{\partial t} + (\vec{v}', \nabla)T' = \chi \Delta T', \\ \frac{\partial \rho'}{\partial t} + \operatorname{div}(\rho' \vec{v}') = 0, \quad \rho' = \rho_0(1 - \beta T'), \quad \beta = -\frac{1}{\rho'}\Big(\frac{\partial \rho'}{\partial T'}\Big)\end{gathered} \tag{1.1}$$

with the boundary conditions

$$\begin{gathered}R_1 v'_r|_{r=R_1} = R_2 v'_r|_{r=R_2} = s, \quad s = \text{const}, \\ v'_\varphi|_{r=R_i} = \Omega_i R_i, \quad v'_z|_{r=R_i} = 0, \quad T'|_{r=R_i} = \Theta_i \quad (i = 1, 2),\end{gathered} \tag{1.2}$$

where $\vec{v}' = \{v'_r, v'_\varphi, v'_z\}$ is the velocity vector, T' is temperature, Π' is pressure, ρ' is the liquid density, ρ_0 is the liquid density for temperature Θ_1, t is time, ν, χ, β are, respectively, the coefficients of kinematic viscosity, thermal conductivity and thermal extension.

The operators Δ and ∇ in the cylindrical coordinates are of the form $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial z^2}$, $\nabla = \{\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}\}$. As the scale of length, velocity, time, temperature and density we take, respectively, R_1, $\Omega_1 R_1$, $1/\Omega_1$, Θ_1 and ρ_0.

The problem (1.1)-(1.2) admits an exact solution [2]:

$$\vec{V}_0=\{v_{0r},v_{0\varphi},0\}\quad v_{0r}=\frac{\varkappa_0}{r},\quad v_{0\varphi}=\begin{cases}ar^{\varkappa+1}+\frac{b}{r}, & \varkappa\neq 2,\\ \frac{a_1\ln r+1}{r}, & \varkappa=-2,\end{cases}$$
$$T_0=c_1+c_2r^{\varkappa P},\quad \Pi_0=\int_1^r\frac{v_{0\varphi}^2}{r}[1-\beta\Theta_1c_2(r^{\varkappa Pr}-1)]dr, \tag{1.3}$$

where

$$a=\frac{\Omega R^2-1}{R^{\varkappa+2}-1},\quad b=1-a;\quad \Omega=\frac{\Omega_2}{\Omega_1},\quad R=\frac{R_2}{R_1},\quad \Theta=\frac{\Theta_2}{\Theta_1},$$
$$a_1=\frac{\Omega R^2-1}{\ln R},\quad c_2=\frac{1-\Theta}{1-R^{\varkappa Pr}};\quad c_1=1-c_2,\quad \varkappa_0=\frac{s}{\Omega_1R_1^2},$$

$Pr=\frac{\nu}{\chi}$ is the Prandtl's number, $\varkappa=\frac{s}{\nu}$ - Reynold's radial number (for $\varkappa>0$ the liquid flows through the inner cylinder, while for $\varkappa<0$ through the outer one).

The solution (1.3) represents stationary flow of the heat-conducting liquid between the rotating permeable cylinders in the presence of the radial temperature gradient in the Boussinesq approximation ([1–3]). This flow is defined by means of the parameters R, Ω, Θ, Pr and independent on the Reynold's number $\mathrm{Re}=\frac{\Omega_1R_1^2}{\nu}$. Many works of theoretical and experimental character are devoted to the investigation of the influence of temperature gradient on the stability of flow between rigid rotating cylinders (nonisothermal Couette flow). At present, the problem dealing with the first loss of stability in that flow and with branching the secondary regimes can be considered as completely studied (see [4]–[7]).

2. A solution of the problem (1.1), (1.2) is sought in the form

$$\vec{v}'=\vec{V}_0+\vec{v}(v_r,v_\varphi,v_z),\quad T'=T_0+c_2PrT,\quad \Pi'=\Pi_0+\Pi/\mathrm{Re}\,. \tag{2.1}$$

Substituting (2.1) into (1.1), (1.2), we arrive at the following problem of finding the perturbations $\vec{v}$, T and Π:

$$\begin{gathered}\frac{\partial v_r}{\partial t}+\omega_1\frac{\partial v_r}{\partial\varphi}+(\vec{V},\nabla)v_r-\frac{v_\varphi^2}{r}+\frac{\varkappa_0}{r}\Big(\frac{\partial v_r}{\partial r}-\frac{v_r}{r}\Big)+\frac{1}{\mathrm{Re}}\frac{\partial\Pi}{\partial r}=\\ =\frac{1}{\mathrm{Re}}\Big(\Delta v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\varphi}{\partial\varphi}\Big)+2\omega_1v_\varphi-\mathrm{Ra}\,\omega_2T\\ \frac{\partial v_\varphi}{\partial t}+\omega_1\frac{\partial v_\varphi}{\partial\varphi}+(\vec{V},\nabla)v_\varphi+\frac{v_rv_\varphi}{r}+\frac{\varkappa_0}{r}\Big(\frac{\partial v_\varphi}{\partial r}+\frac{v_\varphi}{r}\Big)+\frac{1}{\mathrm{Re}\,r}\frac{\partial\Pi}{\partial\varphi}=\\ =\frac{1}{\mathrm{Re}}\Big(\Delta v_\varphi-\frac{v_\varphi}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\varphi}\Big)+g_1v_r\\ \frac{\partial v_z}{\partial t}+\omega_1\frac{\partial v_z}{\partial\varphi}+(\vec{V},\nabla)v_z+\frac{\varkappa_0}{r}\frac{\partial v_z}{\partial r}+\frac{1}{\mathrm{Re}}\frac{\partial\Pi}{\partial z}=\frac{1}{\mathrm{Re}}\Delta v_z\\ \frac{\partial T}{\partial t}+\omega_1\frac{\partial T}{\partial\varphi}+(\vec{V},\nabla)T+\frac{\varkappa_0}{r}\frac{\partial T}{\partial r}=\frac{1}{\lambda P}\Delta T-\frac{g_2}{P_r}v_r\\ \int_0^{2\pi}\int_1^R v_zrdrd\varphi=0,\quad \operatorname{div}\vec{V}=0.\end{gathered} \tag{2.2}$$

$$\vec{v}|_{r=1,R} = 0, \quad T|_{r=1,R} = 0 \tag{2.3}$$

where $\mathrm{Ra} = \beta c_2 \Theta_1 Pr$ is the Rayleigh's number, $\mathrm{Re} = \frac{\Omega_1 R_1^2}{\nu}$ - Reynold's number, $\omega_1 = \frac{v_0 \varphi}{r}$, $\omega_2 = \omega_1^2 r$,

$$g_1(r) = \begin{cases} -(\varkappa + 2) a r^{\varkappa}, & \varkappa \neq 2, \\ -\frac{a_1}{r^2}, & \varkappa = -2, \end{cases} \quad g_2(r) = \varkappa r^{\varkappa Pr - 1}.$$

Assume that the perturbations of velocity $\vec{v}$, temperature T and density are infinitesimal. In (2.2), we omit the nonlinear terms and obtain a linearized problem of stability.

Finding a solution of the linearized problem in the form

$$\vec{v}(v_r, v_\varphi, v_z) = e^{ict}\{u(r), v(r), w(r), \tau(r)\} e^{-i(m\varphi + \alpha z)}, \quad \Pi = q(r) e^{ict} e^{i(m\varphi + \alpha z)},$$

after separation of variables we obtain the spectral problem for the system of ordinary differential equations in the case of axisymmetric stationary ($c = m = 0$) and oscillatory three-dimensional perturbations, where c is an the unknown cyclic frequency (phase velocity of azimuthal waves), α,m– axial and azimuthal wave numbers, respectively.

These problems are reduced to the Cauchy problems for eight ordinary differential equations of the first kind with real and complex coefficients. The use is made of the shooting method together with the Newton's method. For numerical integration of the Cauchi problems we used the standard Runge-Kutta's method. The numerical minimization of Reynolds number Re was performed with respect to the wave numbers m and α. As a result of our calculations, we obtained dependencies from the Reyleigh's number Ra, to critical values of Reynold's number Re and frequency c (phase velocity of azimuthal waves) which corresponds to bifurcations emerging under axisymmetric flows ($m = 0$) and azimuthal waves ($m \neq 0$).

The calculations were performed for the case $R = 2$, $Pr = 0.71$ (working environment is air), for $-10 < \varkappa < 10$ under the rotation of the inner cylinder ($\Omega = 0$), as well as under the rotation of cylinders in the same direction ($\Omega = 0.2$). In Figs. 1 and 2 we can see the dependence of the minimized critical Reynold's number Re on the radial Reynold's number $\varkappa$ for different values Ra.

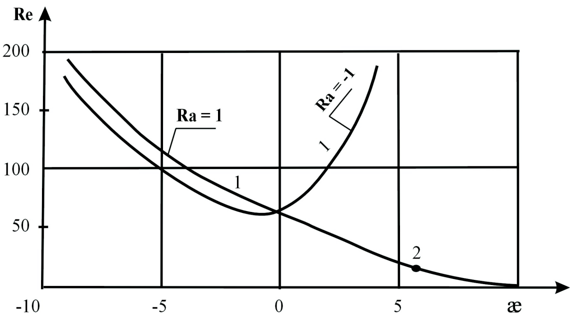

Fig. 1. $\Omega = 0$

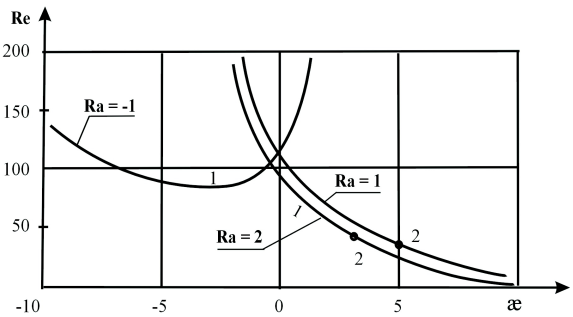

Fig. 2. $\Omega = 0.2$

The segments of curves on which m remains constant, are denoted by $1, 2$; they correspond to axisymmetric stationary ($m = 0$) and oscillatory ($m = 1$) perturbations.

As our calculations show, for $\Omega = 0$ (Fig. 1) when $\mathrm{Ra} < 0$ (temperature of the inner cylinder exceeds that of the outer one) the first instability corresponds to axisymmetric disturbances. Stabilization of the flow (1.3) is observed as the intensity of the outward and inward radial flow increases. Note, that stabilizing effect is stronger for positive $\varkappa$. For $\mathrm{Ra} > 0$ the stabilisation of flow is observed for $\varkappa < 0$, while for $\varkappa > 0$ and increasing $\varkappa$ we obtain a destabilizing effect, moreover for $\varkappa = 7$ there takes place transition from the axisymmetric mode to non-axisymmetric oscillatory modes of period 2π in the azimuthal direction.

When the cylinder rotate in one and the same direction (Fig.2) for $\mathrm{Ra} < 0$ the stabilization of the main flow (1.3) is observed too, while for $\mathrm{Ra} > 0$ as $\varkappa$ increases, we obtain a destabilizing effect, which increases together with the increase of $\varkappa$ and moreover for $\mathrm{Ra} = 1$, $\varkappa = 5$ and $\mathrm{Ra} = 2$, $\varkappa = 3.5$ there arise autooscillations, periodic in azimuthal direction with period 2π. As is seen from Fig.2, when Ra grows the critical number Re decreases.

Thus unlike the flow without temperature gradient between two rotating permeable cylinders [8] we obtain, that positive Raleigh number (temperature of the outer cylinders exceeds that of the inner one) has a destabilizing effect, which increases together with increase Rayleigh number Ra.

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