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A NUMERICAL SOLUTION OF STRING OSCILLATION EQUATION

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Abstract. We have drawn numerical algorithm for Kirchhoff's integro- differential equation that describes the string oscillation. The algorithm has been approved by tests and the results of recounts are represented in the graphics.

Keywords and phrases: Kirchhoff string wave equation, Galerkin's method, Crank-Nicolson difference scheme, Picard iteration process, calculations results.

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1. Statement of the problem. Let us consider the nonlinear equation

$$w_{tt}(x,t)-\left(\lambda+\frac{2}{\pi}\int\limits_0^\pi w_x^2(x,t)dx\right)w_{xx}(x,t)+f(x,t)=0, \tag{1}$$

$$0<x<L,\ \ 0<t\le T,$$

with the initial boundary conditions

$$w(x,0)=w^0(x), w_t(x,0)=w^1(x), \tag{2}$$

$$w(0,t)=w(\pi,t)=0, \tag{3}$$

$$0\le x\le\pi,\ \ 0\le t\le T,$$

where $\lambda>0$, T are given constants and $f(x,t), w^0(x), w^1(x)$ are the given functions.

The equation (1.1) was suggested by Kirchhoff [1] in 1876 as the more precise model of the string's oscillation as compared with D'Alembert's equation $w_{tt}=c^2w_{xx}$. Many authors have investigated this equation in case when $f(x,t)=0$ and its natural generalizations mainly from the point of view of the possibility of solving the equation (1.1). See, for example works by A. Arosio, S. Bernstein, P. D'Ancona, R. Narasimha, K. Nishihara, S. Panizzi, S.I. Pohozaev, S. Spagnolo. Several works have been done in the field of studying the approximate methods for solving the equation (1.1). See, for example works by F. Attugui, I. Christie, R.W. Dickey, I.S. Liu, J. Peradze, M.A. Rincon, J.M. Sanz-Serna. Here we are considering one of the numeral algorithm [2] of the approximate solution of the equation (1.1) and present the results of counting.

2. The Algorithm. The algorithm consists of three parts.

The first part - the Galerkin method. An approximate solution of problem (1.1) - (1.3) is written in the form

$$w_n(x,t)=\sum_{i=1}^n w_{ni}(t)\,\sin ix, \tag{1}$$

where the coefficients $w_{ni}(t)$ are defined by the Galerkin method from the system of nonlinear differential equations

$$w''_{ni}(t) + \left(\lambda + \sum_{j=1}^{n} j^2 w_{nj}^2(t)\right) i^2 w_{ni}(t) + f_i(t) = 0, \tag{2}$$

$$0 < t \le T,$$

with the conditions

$$w_{ni}(0) = a_i^0,\ w'_{ni}(0) = a_i^1,\ i = 1, 2, \dots, n, \tag{3}$$

where

$$f_i(t) = \frac{2}{\pi}\int_0^{\pi} f(x,t)\sin ix dx,\quad a_i^{(p)} = \frac{2}{\pi}\int_0^{\pi} w^p(x)\sin ix dx,\quad p = 0, 1.$$

We introduce the functions

$$u_{ni}(t) = w'_{ni}(t),\quad v_{ni}(t) = iw_{ni}(t),\ i = 1, 2, \dots, n, \tag{4}$$

and replace system (2.2), (2.3) by an equivalent system of the first order

$$u'_{ni}(t) + \left(\lambda + \sum_{j=1}^{n} v_{nj}^2(t)\right) iv_{ni}(t) + f_i(t) = 0, \tag{5}$$

$$v'_{ni}(t) = iu_{ni}(t),\ 0 < t < T, \tag{6}$$

$$u_{ni}(0) = a_i^1,\ \ v_{ni}(0) = ia_i^0,\ \ i = 1, 2, \dots, n. \tag{7}$$

The second part - the Crank-Nicolson type difference scheme. We proceed to solve problem (2.5)-(2.7) by means of the difference method. On the time interval $[0, T]$ let us introduce the grid $\{t_m | 0 = t_0 < t_1 < \cdots < t_m = T\}$ with a generally variable step $\tau_m = t_m - t_{m-1}, m = 1, 2, \dots, M$. Let us use notation $f_i^m = f_i(t_m)$. Approximate values of $u_{ni}(t)$ and $v_{ni}(t)$ on the m-th time layer, i.e. for $t = t_m,\ m = 0, 1, \dots, M$, denoted by u_{ni}^m and v_{ni}^m are defined by the implicit symmetric scheme

$$\frac{u_{ni}^m - u_{ni}^{m-1}}{\tau_m} + \left\{\lambda + \frac{1}{2}\left[\sum_{j=1}^{n}((v_{nj}^m)^2 + (v_{nj}^{m-1})^2)\right]\right\}\frac{v_{ni}^m + v_{ni}^{m-1}}{2} + f_i^m = 0 \tag{8}$$

$$\frac{v_{ni}^m - v_{ni}^{m-1}}{\tau_m} = i\frac{u_{ni}^m + u_{ni}^{m-1}}{2},\ \ m = 1, 2, \cdots, M,\ \ i = 1, 2, \cdots, n, \tag{9}$$

$$u_{ni}^0 = a_1^1,\quad u_{ni}^0 = ia_i^0. \tag{10}$$

The third part - the Picard type iteration process. To solve the system of nonlinear equations (2.8)-(2.10) we assumed that the counting is performed layer wise by iteration. After getting a solution on the $(m-1)$-th layer, we process to the m-th layer.

Denote by $u_{ni}^{m,k}$ and $v_{ni}^{m,k}$ the k-th iteration approximation of u_{ni}^m and v_{ni}^m, $k = 0, 1, \ldots$. Let us use the following iteration method

$$u_{ni}^{m,k} = u_{ni}^{m-1,k_0} - \frac{\tau_m i}{2}\left\{\lambda + \frac{1}{2}\left[\sum_{j=1}^{n}((v_{nj}^{m,k-1})^2 + (v_{nj}^{m-1,k_o})^2)\right]\right\}\times$$

$$\times(v_{nj}^{m,k-1} + v_{nj}^{m-1,k_0}) - \tau_m f_i^m, \tag{11}$$

$$v_{ni}^{m,k} = v_{ni}^{m-1,k_0} + \frac{\tau_m i}{2}(u_{ni}^{m,k-1} + u_{ni}^{m-1,k_0}), \tag{12}$$

k_0 is the amount of carry out iteration in $m-1$ level.

We calculate the components $u_{ni}^{m,k}$ and $v_{ni}^{m,k}$, by formulas (2.11), (2.12). Then, for chosen n and for $t = t_m$, the series

$$\sum_{i=1}^{n} w_{ni}^{m,k} \sin ix, \tag{13}$$

where $w_{ni}^{m,k} = \frac{1}{i} v_{ni}^{m,k}$, gives at the k-th iteration step, an approximate value of the exact solution $w(x, t_m)$ of the problem (1.1)-(1.3).

3. Algorithms Realization. The algorithm proposed in subsection 2 enables us to find approximate solutions of problem (1.1)-(1.3). The approximate program has been designed in Turbo Pascal algorithm language and calculations have been done on the computer. The results obtained are good enough. The algorithm has been approved by tests and the results of recounts are represented in the graphics. A problem (1.1)-(1.3) with the following data is discussed:

$$f(x,t) = -6t\sin 2x + (\lambda + 1 + 4t^6)(\sin x + 4t^3 \sin 2x),$$

$$w^0(x) = \sin x,\ w^1(x) = 0,\ \lambda = 0.4,\ T = 1.$$

Corresponding exact solution is $w(x,t) = \sin x + t^3 \sin 2x$.

Calculations have been done for $n = 5, M = 20, \tau = 0.05$.

The amount of iteration on every time level $k_0 = 20$.

Figure 1 corresponds to the exact solution and Figure 2 - to the solution according to (2.13) formula.

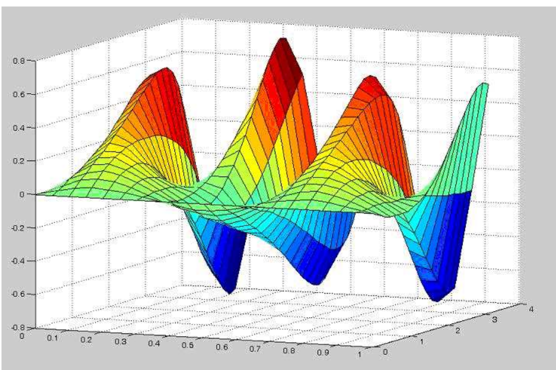

Fig.1.

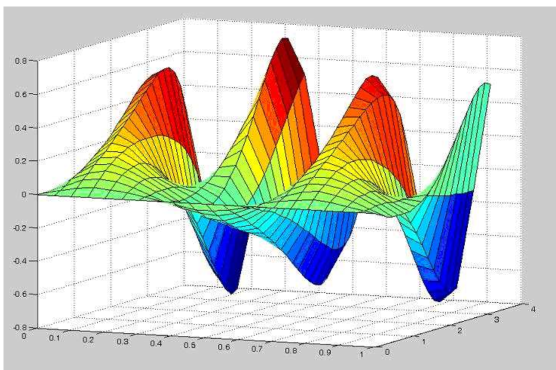

Fig.2.

The comparison of the graphics shows that the approximate solution only slightly differs from the exact solution.

R E F E R E N C E S

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