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ON APPLICATION OF ALTERNATING TO PERTURBATION TECHNIQUES METHOD TO SINGULAR INTEGRAL EQUATIONS CONTAINING AN IMMOVABLE SINGULARITY

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Abstract. Problems of approximate solution of some linear non-homogeneous operator equation is studied with an approach alternative to asymptotic method. Our alternative method is based on representation of unknown vector over the small parameter with orthogonal series instead of asymptotic one. In such a case system of three-point operator equations of special structure in received. For system solving a certain regular method is used. On the basis of the suggested method the programming production is created and realized by means of computer. Algorithms and program products represent a new technology of approximate solving of some singular integral equations containing an immovable singularity.

Keywords and phrases: Non-homogeneous operator equation, orthogonal series, asymptotic method, alternative method.

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1. Introduction. The article [1] deals with the approximate solution of some linear non-homogeneous operator equation using an alternative method of an asymptotic method. Algorithms of approximate solution of a linear non-homogeneous operator equation are studied in the article [2]. Both asymptotic and alternative methods are used for this. In the same article the above-mentioned methods are approbated on the double-point boundary problems, while in article [3] on the problems of approximate solution of some integro-differential equation. This article presents an algorithm of an approximate solution of a singular integral equation containing an immovable singularity.

2. Algorithms of Approximate Solution of a Linear Operator Equation. Let us have a non-homogeneous operator equation

$$Lu + \varepsilon Mu = f, \tag{1}$$

where L and M are linear operators in any standardized space. In addition, there exist inverse operators L^{-1} and $(L + \varepsilon M)^{-1}$, where $\varepsilon \in [-1; +1]$ is a small parameter.

Represent the solution $u(x)$ in the form of Fourier-Legandre series

$$u(x) = \gamma \sum_{k=0}^{\infty} \varepsilon^k \nu_k(x) + (1-\gamma) \sum_{k=0}^{\infty} P_k(\varepsilon) w_k(x),$$

where $\{P_k(\varepsilon)\}$ is a system of Legendre polynomials, $w_k(x)$ and $\nu_k(x)$ are unknown coefficients, γ is numerical parameter.

In case $\gamma = 1$ we have an asymptotic method (Poincare-Lyapunov's method)

$$u(x) = \sum_{k=0}^{\infty} \varepsilon^k \nu_k(x) \tag{2}$$

By putting series (2) into equation (1) and equating coefficientsof terms with the same degrees of ε, we get a system of two-point recurrence operator equations having the following form:

$$\begin{cases} L\nu_0 = f_0, \\ L\nu_k = -M\nu_{k-1} + f_k, \qquad k = 1, 2, 3, \dots. \end{cases} \tag{3}$$

When $\gamma = 0$, we have an approach alternative to the asymptotic method

$$u(x) = \sum_{k=0}^{\infty} P_k(\varepsilon) w_k(x) \tag{4}$$

If set series (4) into equation (1) use the main properties of Legandre polynomial and equate coefficients with equal degrees of ε, we shall get a system of three-point recurrence operator equations of the following form:

$$\begin{cases} Lw_0 + \dfrac{1}{3} M w_1 = f_0, \\ Lw_k + \dfrac{k}{2k-1} M w_{k-1} + \dfrac{k+1}{2k+3} M w_{k+1} = f_k, \qquad k = 1, 2, 3, \dots \end{cases} \tag{5}$$

Suppose that the right hand side of equations f does not depend on small parameter ε therefore $f_k = 0,\ k = 1, 2, 3, \dots$, and even it depends on ε, this fact has no essential influence on realization of computing schemes.

Solutions of system of double-point operator equations (3) have the following simple form:

$$\nu_0 = L^{-1} f_0,$$
$$\nu_i = (-1)^i L^{-1} M \nu_{i-1}, \quad i = 1, 2, 3, \dots,$$

Instead of infinite system of three-point operator equations (5), let us take its finite part, in addition even number of equations ($N = 2n,\ n \in N$).

The obtained system of equations may be decomposed into two subsystems, We find w_k, by means of regular process [1].

For clearness, let us consider special cases, when $N = 2$ and $N = 4$. When $N = 2$, we get $w_0 = \nu_0, w_1 = \nu_1$, and approximate solution of equation (1) has a form:

$$u_1(x) = w_0(x) + \varepsilon w_1(x).$$

When $N = 4$ then we have $w_0 = \nu_0 + \frac{1}{3}\nu_2,\ w_2 = \frac{2}{3}\nu_2, w_1 = \nu_1 + \frac{3}{5}\nu_3,\ w_3 = \frac{2}{5}\nu_3$, and approximate solution of equation (1) has a form

$$u_3 = w_0(x) + \varepsilon w_1(x) + P_2(\varepsilon) w_2(x) + P_3(\varepsilon) w_3(x),$$

where $P_2(\varepsilon) = \frac{1}{2}(3\varepsilon^2 - 1)$, $P_3(\varepsilon) = \frac{1}{2}(5\varepsilon^3 - 3\varepsilon)$.

The above-mentioned method has been approved for approximate solution of double-point linear boundary value problems [2], some linear non-homogeneous integro-differential equations [3] and singular integral equations containing an immovable singularity.

3. Approximate Solving of Some Singular Integral Equations. Let us consider the following singular integral equation containing an immovable singularity

$$\begin{cases} \frac{1}{\pi}\int\limits_0^1 \left[\frac{1}{t-x} + \frac{\varepsilon}{t+x}\right] u(t)dt = f(x), \quad x \in [0,1], \\ \int\limits_0^1 u(t)dt = 0, \end{cases} \tag{6}$$

where $u(t) \in H^*([0,1]), \varepsilon \in [-1,1], f(x) \in H_\mu[0,1], \qquad 0 < \mu \leq 1$.

Analysis of the above-mentioned integral equation and study of their exact and approximate solving methods are accompanied with some specific complexities due to the fact, that the solution has a composite asymptotic, which can be considered only in certain cases introducing weight functions. If we have square root type singularity on both ends of the integral, then Chebishev orthogonal polynomials can be used. The solution of equation (6) we can represent in the following form :

$$u(t) = u_0(t)/t^\alpha\sqrt{1-t},$$

where $u_0(t) \in H_0([0,1])$, $u_0(0) = 0$, α depends on material elasticity constants, $0 < \alpha < 1$.

Integral equation (6) can be solved by three approximate methods: spectral, collocation and asymptotic ones. In the work [4] we use the collocation method.

In the present paper, boundary problem (10) is solved by asymptotic method and by an approach, alternative to the asymptotic one. In our case the basic operator is $Lu = \frac{1}{\pi}\int\limits_0^1 \frac{1}{t-x}u(t)dt$ and operator describing the perturbation degree is $Mu = \int\limits_0^1 \frac{1}{t+x}u(t)dt$.

Let us discuss a case when there is a square root type singularity on each end of the interval. To invert main operators we use well -known formula [5]. Analogously, [3] here, in the case of asymptotic method the approximation formula of the inverted operator are constructed: and operator describing the perturbation degree is

$$\nu_0(x) = \frac{\pi}{\sqrt{x(1-x)}}\int\limits_0^1 \frac{\sqrt{t(1-t)}}{t-x}f(t)dt,$$

$$\nu_i(x) = \frac{\pi}{\sqrt{x(1-x)}}\int\limits_0^1 \frac{\sqrt{t(1-t)}}{t-x}\int\limits_0^1 \frac{1}{t+\xi}\nu_{i-1}(\xi)d\xi dt,$$

$$i = 1, 2, 3...$$

4. Numerical Experiments. The algorithm proposed in subsection 2 enables us to find approximate solutions of problem (6) for $N = 2$ and $N = 4$ as by the asymptotic method so by the alternative approach. To calculate multiple integrals with predetermined exactness we use Simpson's quadrature formula.

For approximate solving boundary value problem the complex of programs in algorithm language *Turbo Pascal* is composed and many numerical experiments are carried out. The results obtained are good enough. In practical tasks it is often sufficient if we take in the case of applying asymptotic and alternative methods$N \leq 4$ is quite enough.

Algorithms and program products represent a new technology of approximate solving of some singular integral equations containing an immovable singularity.

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