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VARIATIONAL FORMULATION OF ONE NONLOCAL BOUNDARY PROBLEM

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Abstract. One nonlocal problem for second order ordinary differential equation with integral type nonlocal boundary condition is considered. Variational formulation by using inner product constructed by symmetric continuation of a function is studied.

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Following nonlocal boundary value problem is considered: Let us find a function $u(x) \in C^{(2)}\ ]-a,\ 0[\cap C[-a,\ 0]$ for which the second order ordinary differential equation with integral type nonlocal boundary condition are satisfied:

$$-(k(x)u'(x))' + q(x)u(x) = f(x), \quad x \in]-a,\ 0[, \tag{1}$$

$$u(-a) = 0, \tag{2}$$

$$\int\limits_{-\xi}^{0} k(x)u'(x)dx = 0, \tag{3}$$

where $\xi \in]0,\ a[$ is a fixed point, $f(x) \in C\,[-a,\ 0]$, $q(x) \in C[-a,\ 0]$, $k(x) \in C^{(1)}[-a,\ 0]$, $k(x) \geq k_0 > 0$, $q(x) \geq 0$ for $x \in [-a,\ -\xi]$ and $q(x) \equiv 0$ for $x \in [-\xi,\ 0]$.

Note, that when the function $k(x)$ is constant, expression (3) presents Bitsadze-Samarskii nonlocal boundary condition [1].

Many scientific works are devoted to the investigation of nonlocal problems (see, for example, [1]-[12]).

It is known how great role takes place variational formulation for investigation of boundary problems. In nonlocal boundary value problems this task is in the beginning of study (see, for example, [13]-[18]).

The aim of the present note is to state and study variational formulation of problem (1)-(3).

We denote by $D[-a, 0]$ a lineal of all real functions such that each of its functions $v(x)$ be defined a.e. on $[-a, 0]$, $|v(0)| < +\infty$ and $v(x) \in L_2[-a,\ 0]$.

Note that, to give a function $u(x) \in D\,[-a,\ 0]$, one should essentially specify the pair $(v(x), v(0))$ $(x \in [-a,\ 0[)$. Functions $v_1(x)$ and $v_2(x)$ are the same elements of the lineal $D[-a,\ 0]$ if $v_1(x) = v_2(x)$ a.e. on $[-a,\ 0[$ and $v_1(0) = v_2(0)$.

On the lineal $D[-a,\ 0]$ we define operator τ of symmetric continuation as follows:

$$\tau v(x) = \begin{cases} v(x), & \text{for} \quad x \in [-a,\ 0], \\ -v(-x) + 2v(0), & \text{for} \quad x \in]0,\ \xi]. \end{cases}$$

Function $\tilde{v}(x) = \tau v(x)$ defined a.e. on the $[-a,\, \xi]$ is corresponded to each function $v(x)$ and function $\tilde{v}(x) - v(0)$ is odd a.e. on the $[-\xi,\, \xi]$.

Let us define following inner product on lineal $D[-a, 0]$

$$[u,\, v] = \int\limits_{-\xi}^{\xi} \int\limits_{-a}^{x} \tilde{u}(s)\tilde{v}(s)dsdx. \tag{4}$$

By the inner product (4) the lineal $D[-a,\, 0]$ becomes pre-Hilbert space which we denote via $H[-a,\, 0]$. For the norm produced by inner product (4) we use the notation

$$||v||_H = \left(\int\limits_{-\xi}^{\xi} \int\limits_{-a}^{x} \tilde{v}^2(s)dsdx \right)^{1/2}.$$

Theorem 1. *The norm defined on the lineal $D[-a,\, 0]$ by the equality*

$$||v|| = \left(||v||^2_{L_2} + v^2(0)\right)^{1/2},$$

is equivalent to the norm $||\cdot||_H$, where $||v||^2_{L_2} = \int\limits_{-a}^{0} v^2(x)dx$.

Let $D_A[-a,\, 0]$ lineal of the functions of the space $H[-a,\, 0]$ be a domain of definition of the operator $Au = -(ku')' + qu$. For each function $v(x)$ of the lineal $D_A[-a,\, 0]$ the following conditions are fulfilled:

$$v(x) \in C^{(2)}[-a,\, 0],\ v(-a) = 0,\ v'(0) = 0,\ v''(0) = 0,$$

$$\int\limits_{-\xi}^{0} k(x)u'(x)dx = 0.$$

Theorem 2. *The lineal $D_A[-a,\, 0]$ is dense in $H[-a,\, 0]$.*

Thus, an operator A acts from the lineal $D_A[-a,\, 0]$ into the $H[-a,\, 0]$.

Theorem 3. *The operator A is symmetric on the lineal $D_A[-a,\, 0]$.*

Theorem 4. *The operator A is positively defined on the $D_A[-a,\, 0]$.*

Thus, A is an operator defined positively on the dense lineal $D_A[-a,\, 0]$ in the Hilbert space $H[-a, 0]$. Follow the standard way [19]. Onto the lineal $D_A[-a,\, 0]$, let us introduce a new inner product

$$[u, v]_A = [Au, v] = \int\limits_{-\xi}^{\xi} \int\limits_{-a}^{x} \left(\overline{k}(s)\tilde{u}'(s)\tilde{v}'(s) + \widetilde{q}(s)\widetilde{u}(s)\widetilde{v}(s)\right) dsdx. \tag{5}$$

By inner product (5) the lineal $D_A[-a,\, 0]$ is transformed into the pre-Hilbert space. Denote it via $S_A[-a,\, 0]$. Complete this space with the norm correspondent to the (5) inner product. This norm, as is easy to show, is equivalent to the norm defined by the equality

$$|||u|||^2 = ||u||^2_{W^1_2} + u^2(0), \tag{6}$$

where $\|\cdot\|_{W_2^1}$ is the norm of the usual Sobolev space W_2^1.

Denote with $H_A[-a,\,0]$ a Hilbert space obtained as a result of completing with respect to norm (6). The space consists in those functions of $W_2^1[-a,\,0]$, which satisfy the conditions (2) and (3).

Let $\alpha \in R$. Consider a pair $(f(x),\,\alpha)$. It defines the unique function $f_\alpha(x)$ of the space $H[-a,\,0]$. For each such function a functional

$$F_\alpha(v) = [v, v]_A - 2[f_\alpha, v] \tag{7}$$

has unique minimizing function $u_\alpha(x) \in H_A[-a,\,0]$ which satisfies the relation

$$[u_\alpha, v]_A = [f_\alpha, v]$$

for all $v(x) \in H_A[-a, 0]$.

As is easy to see,

$$u_\alpha(x) = u_0(x) + \alpha\omega(x),$$

where $\omega(x)$ is a minimizing function of the functional (7) in that case when the first term of the pair $(f(x),\,\alpha)$ is identical to zero function on the $[-a,\,0[$, and $\alpha = 1$.

Theorem 5. *Let $u(x)$ be a solution of problem (1)-(3). Then it is a minimizing function of the functional $F_0(v)$ in the space $H[-a,\,0]$.*

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