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A NON-LOCAL BOUNDARY PROBLEM FOR EQUATION $\Delta^k v = 0$ ($k \leq 3$) IN THE SPACE $C^{(2k,\alpha)}(\overline{\Omega})$

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Abstract. Non-local boundary problems for equations $\Delta^2 v = 0$ and $\Delta^3 v = 0$ in the space $C^{(2k,\alpha)}(\overline{\Omega})$ $k \leq 3$ are considered, Ω is simply connected bounded domain from the class $C^{(2k,\alpha)}$, $0 < \alpha \leq 1$; S - closed surface from $C^{(2k,\alpha)}$, $S \subset \Omega$; $\zeta = z(x)$ - $C^{(2k,\alpha)}$-diffeomorphism from $\partial\Omega$ into S.

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Let us consider the non-local boundary problem for the equation $\Delta^2 v = 0$ in the space $C^{(4,\alpha)}(\overline{\Omega})$. Let us find the solution to the equation $\Delta^2 v = 0$ satisfying the boundary values

$$\begin{aligned} & v(x) - v(z(x)) = f(x), \ x \in \partial\Omega, \ f \in C^{(4,\alpha')}, \\ & \frac{\partial v(x)}{\partial \nu_x} = g(x), \quad x \in \partial\Omega, \ g \in C^{(3,\alpha')}, \quad 0 < \alpha < \alpha' < 1, \end{aligned} \tag{1}$$

where ν_x is the unit vector of the outer normal to $\partial\Omega$ at the point x.

Non-local boundary problems are considered in [1-3].

For smooth domains the non-local boundary problem in R^3 for the Laplace equation is stated as follows: let Ω be a simply connected bounded domain from the class $C^{(2,\alpha)}$, $0 < \alpha < 1$, S be a closed surface from $C^{(2,\alpha)}$, $S \subset \Omega$. Let further $\zeta = z(x)$ be a $C^{(2,\alpha)}$-diffeomorphism [4] from $\partial\Omega$ into S. Assume that the boundary function $f \in C(\partial\Omega)$.

We will find a function $\varphi \in C(\partial\Omega)$ for which the following boundary condition is satisfied

$$\varphi(x) - K\varphi(x) = f(x), \quad x \in \partial\Omega,$$

where

$$K\varphi(x) = v(z(x)) = -\int\limits_{\partial\Omega} \frac{\partial G(z(x), y)}{\partial \nu_y} \varphi(y) dS_y.$$

Here G is the Green function of the Dirichlet problem for the domain Ω, ν_y is the outer normal, $x \in \partial\Omega$.

The non-local boundary problem in the disc is studied in the monograph [5, p. 312].

Let us introduce necessary definitions.

The Newton volume potentials and simple-layer potentials are defined as follows:

$$V^g(x) = \int\limits_{\Omega} \Gamma(x,y) g(y) dy, \quad U^{\Psi}(x) = \int\limits_{\partial\Omega} \Gamma(x,y) \Psi(y) dSy,$$

where $g \in C(\overline{\Omega}), \Psi \in C(\partial\Omega)$, $\Gamma(x,y) = |x-y|^{-1}$. Let us note that

$$G(x,y) = \Gamma(x,y) - U^{\delta'_x}(y),$$

where δ'_x is the density of balayage [6] for the unit Dirac measure δ_x, $x \in \Omega$,

$$\Gamma(x,y) = U^{\delta'_x}(y), \quad y \in R^3 - \Omega, \ \delta'_x \in C(\partial\Omega).$$

Let us define the balayage operator [6]

$$f'(y) = Tf(y) = -\int\limits_{\Omega} \frac{\partial G(x,y)}{\partial \nu_y} f(x)dx, \quad f \in C(\overline{\Omega}).$$

Define the space

$$B_2 = \left\{ f : f \in C^{(4,\alpha)}, \ \int\limits_{\partial\Omega} f(x)\gamma(x)dS_x = 0 \right\}, \quad U^\gamma(x) = 1, \ x \in \overline{\Omega}, \ \gamma \in C^{(3,\alpha)}.$$

It is assumed that $\Omega \in C^{(6,\alpha')}$, $S \in C^{(6,\alpha')}$, $\zeta = z(x) \in C^{(6,\alpha')}$.

Theorem 1. The solution to problem (1) exists if and only if $f \in \gamma^\perp$, where $U^\gamma(x) = \int_{\partial\Omega} |x-y|^{-1}\gamma(y)dS_y = 1$, $x \in \Omega$, $\gamma \in C^{(3,\alpha)}$, $\gamma^\perp$ is the annihilator, $f \in C^{(4,\alpha)}$.

Proof of sufficiency. We seek the solution in the space $C^{(4,\alpha)}(\overline{\Omega})$. Obviously the solution v admits the representation

$$v(x) = H_0(x) - \int\limits_{\Omega} G(x,y)H_1(y)dy \tag{2}$$

where H_0, H_1 are harmonic functions for which $H_0(x)=v(x)$, $H_1(x)=\Delta v(x)$, $x \in \partial\Omega$. $H_0(x)= v(x)= \varphi(x)$, $x \in \partial\Omega$ $(v(x)=\varphi(x),\ x \in \partial\Omega)$. Due to boundary conditions (2) gives

$$\frac{\partial v(x)}{\partial \nu_x} = g(x) = \frac{\partial H_0(x)}{\partial \nu_x} + H'_1(x), \ H'_1(x) = g(x) - \frac{\partial H_0(x)}{\partial \nu_x}, \ H_1(x) = T^{-1}H'_1,$$

$$H_0(x) - H_0(z(x)) = f(x) - \int\limits_{\Omega} G(z(x),y)H_1(y)dy, \ H_1(y) = T^{-1}g(y)$$

$$-T^{-1}\left(\frac{\partial H_0}{\partial \nu_x}\right)(y), \ \varphi(x) - K\varphi(x) = f_1(x), \ f_1(x) = f(x) - T^{-1}g(y),$$

$$K\varphi(x) = H_0(z(x)) + \int G(z(x),y)T^{-1}\left(\frac{\partial H_0}{\partial \nu_x}\right)(y)dy \quad (H_0|_{\partial\Omega} = v|_{\partial\Omega} = \varphi).$$

It is not hard to show that the operator $A\varphi = \varphi - K\varphi$ maps B_2 into B_2. Besides the homogeneous equation $\varphi - K\varphi = 0$ in B_2 has the trivial solution only. Thus, in the space B_2 there exists a continous inverse operator A^{-1}.

The proof of necessity follows from the previous argument.

Now we pose the non-local problem for $\Delta^3 v = 0$. Let us find the solution v belonging to the space $C^{(6,\alpha)}(\overline{\Omega})$ satisfying the following boundary conditions:

$$v(x) - v(z(x)) = f(x) \in C^{(6,\alpha')}, \quad 0 < \alpha < \alpha' < 1,$$

$$\frac{\partial v(x)}{\partial \nu_x} = g_1(x) \in C^{(5,\alpha')}, \qquad \Omega \in C^{(8,\alpha')}\ S \in C^{(8,\alpha')}, \tag{3}$$

$$\frac{\partial^2 v(x)}{\partial \nu_x^2} = g_2(x) \in C^{(4,\alpha')}, \qquad \zeta = z(x) \in C^{(8,\alpha')}.$$

The solution $v \in C^{(6,\alpha)}(\overline{\Omega})$ admits the

$$v(x) = H_0(x) - \int\limits_{\Omega} G(x,y)H_1(y)dy + \int\limits_{\Omega} G(x,y) \int\limits_{\Omega} G(y,z)H_2(z)dzdy.$$

Therefore due to the boundary conditions we come to the second kind Fredholm equation

$$\varphi - K\varphi = f_1 \quad (v|_{\partial\Omega} = H_0|_{\partial\Omega} = \varphi),$$

where f_1 depends on f, g_1, g_2, and $K\varphi$ depends on H_0, H_1 and H_2. Besides, H_1 and H_2 are defined through $\varphi, g_1, g_2, \frac{\partial H_0}{\partial \nu_x}, \frac{\partial^2 H_0}{\partial \nu_x^2}$. In particular, to find H_2 we use the first boundary condition

$$\frac{\partial v}{d\nu_x} = g_1 = \frac{\partial H_0}{\partial \nu_x} - H_1' - TV_G^{H_2} \quad \left(V_G^{H_2}(x) = \int\limits_{\Omega} G(x,y)H_2(y)dy\right).$$

Thus,

$$TV_G^{H_2}(x) = \Psi_2(x) = \frac{\partial H_0(x)}{\partial \nu_x} - H_1'(x) - g_1(x). \tag{4}$$

To define the boundary value H_1 one has to consider the second derivative with respect to ν_x of the expression [7, p. 115]

$$v(x) = H_0(x) - \int\limits_{\Omega} G(x,y)F(y)dy \quad F|_{\partial\Omega} = H_1,$$

$$H_1(x) = \frac{1}{4\pi}\left[g_2(x) - \frac{\partial^2 H_0(x)}{\partial \nu_x^2} - \frac{\partial^2 U_i^{\Psi_1}(x)}{\partial \nu_x^2} + \frac{\partial^2 U_e^{\Psi_1}(x)}{\partial \nu_x^2}\right],$$

$$\Psi_1(x) = TF(x) = g_1(x) - \frac{\partial H_0(x)}{\partial \nu_x}.$$

Due to (4) H_2 $(\Delta H_2 = 0)$ is defined by

$$\int\limits_{\Omega} \frac{V_G^{H_2}(x)dx}{|x-y|} = \int\limits_{\partial\Omega} \frac{\Psi_2(x)dS_x}{|x-y|}, \quad y \in R^3 - \Omega.$$

Thus, the following assertion holds.

Theorem 2. *The solution to problem* (3) *exists if and only if* $f \in B_3$, *where*

$$B_3 = \left\{ f : f \in C^{(6,\alpha)}, \int\limits_{\partial\Omega} f(x)\gamma(x)dS_x = 0 \right\}, \ U^{\gamma}(x) = 1, \ x \in \overline{\Omega}, \ \gamma \in C^{(5,\alpha)}(\partial\Omega).$$

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