Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 23, 2009

ON SOME PROPERTIES OF THE SOLUTIONS SPACE OF IRREGULAR CARLEMAN-VEKUA EQUATION

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Abstract. In this paper we investigate such solutions of the irregular Carleman-Vekua equations which satisfy certain additional asymptotic conditions.

Keywords and phrases: Irregular Carleman-Vekua equation, Liouville theorem.

AMS subject classification (2000): 30G20.

We investigate the solution space following first order elliptic partial differential equation

$$\partial_{\overline{z}} w + Aw + B\overline{w} = 0, \tag{1}$$

on complex plane $\mathbb{C}$. Let N be a given non-negative integer. Denote by $\Omega(N)$ the space of all regular solutions of the equation (1) in the complex plane $\mathbb{C}$ satisfying the condition

$$w(z) = O(z^N), \quad z \to \infty.$$

It is clear, that $\Omega(N)$ is an $\mathbb{R}$-linear space and as is known if the coefficients of the equation (1) satisfy the regularity condition

$$A, B \in L_{p,2}(\mathbb{C})$$

on $\mathbb{C}$ when $p > 2$, then (see [1])

$$\dim_{\mathbb{R}} \Omega(N) = 2N + 2.$$

Denote by $J_0(\mathbb{C})$ and $J_1(\mathbb{C})$ the sets of those functions $a \in L_p^{loc}(\mathbb{C})$, $p > 2$, for which there exists a solution $Q(z)$ of the equation (1) satisfying the following conditions

$$ReQ(z) = O(1), \quad z \in \mathbb{C},$$

and

$$z^n \exp\big\{Q(z)\big\} = O(1), \quad z \in \mathbb{C},$$

for every natural number n, respectively (see [2]).

Theorem 1. *Let the coefficients of the equation (1) satisfy the condition*

$$A \in J_1(C), \quad B \in L_{p,2}(\mathbb{C}), \tag{2}$$

where $p > 2$. Then for arbitrary nonnegative integer N we have

$$\dim \Omega(N) = 0.$$

Proof. Let $w \in \Omega(N)$, and $Q(z)$ be the $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $A(z)$, which participates in the definition of the class $J_1(\mathbb{C})$. Since w satisfies the condition

$$w(z) = O(z^N), \quad z \to \infty,$$

the function $w^*(z)$, defined by formula

$$w^*(z) = w(z)\, e^{Q(z)}, \quad z \in \mathbb{C}, \tag{3}$$

is a regular solution of the equation

$$\frac{\partial w^*}{\partial \overline{z}} + B^* \overline{w^*} = 0 \tag{4}$$

where $B^*(z) = B(z)\, e^{2iImQ(z)}$ and

$$w^*(z)\, e^{-Q(z)} = O(z^N), \quad z \to \infty. \tag{5}$$

The condition (5) implies, that

$$w^*(z) = O\big(z^N\, e^{Q(z)}\big), \quad z \to \infty. \tag{6}$$

Since the function $Q(z)$ is the $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $A(z)$, one has

$$\lim_{z \to \infty} z^N\, e^{Q(z)} = 0. \tag{7}$$

It follows from (6) and (7), that

$$\lim_{z \to \infty} w^*(z) = 0, \tag{8}$$

i. e. $w^*(z)$ is a regular solution of the regular equation (4). Since the condition (8) is fulfilled, one has by means of the Liouville theorem, concerning the generalized analytic functions, that $w^*(z) \equiv 0$ for arbitrary $z \in \mathbb{C}$. Then from the equality (3) one has $w(z) \equiv 0$ for arbitrary $z \in \mathbb{C}$. Therefore $\dim \Omega(N) = 0$.

Theorem 2. *Let the coefficients of the equation (1) satisfy the condition*

$$-A \in J_1(C), B \in L_{p,2}(\mathbb{C}),$$

$p > 2$. *Then for arbitrary nonnegative integer* N

$$\dim \Omega(N) = \infty.$$

Proof. Let $w \in \Omega(N)$ and let $-Q(z)$ be the $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $-A(z)$, which participates in the definition of the class $J_1(\mathbb{C})$. Then the function $w^*(z)$ defined by the formula (3) is a regular solution of the equation (4) and satisfies (6).

On the contrary, if $w^*(z)$ is a regular solution of the regular equation (4) and satisfies (6), then the function $w(z)$ defined by the formula (3) is a regular solution of the equation (1) and satisfies (2), i.e. $w \in \Omega(N)$. Since the function w^* is a regular solution of the regular equation (4), there exists an entire function $\Phi^*(z)$, such that the condition:

$$w^*(z) = \Phi^*(z)\, e^{-T_{\mathbb{C}}(B^* \frac{\overline{w^*}}{w^*})(z)} \tag{9}$$

is fulfilled.

We conclude from the conditions (6) and (9), that

$$\Phi^*(z)\, e^{-T_{\mathbb{C}}(B^* \frac{\overline{w^*}}{w^*})(z)} = O\big(z^N\, e^{Q(z)}\big). \tag{10}$$

It follows from (10), that

$$\Phi^*(z) = O\big(z^N\, e^{Q(z)}\, e^{T_{\mathbb{C}}(B^* \frac{\overline{w^*}}{w^*})(z)}\big), \quad z \to \infty. \tag{11}$$

Since $B^* \in L_{p,2}(\mathbb{C})$, $p > 2$, and the function $\dfrac{\overline{w^*}}{w^*}$ is bounded on the whole plane, one has $B^* \frac{\overline{w^*}}{w^*} \in L_{p,2}(\mathbb{C})$, $p > 2$. It means that the function $T_{\mathbb{C}}\Big(B^* \dfrac{\overline{w^*}}{w^*}\Big)(z)$ is bounded on the whole plane $\mathbb{C}$. Therefore it follows from (11), that

$$\Phi^*(z) = O\big(z^N\, e^{Q(z)}\big), \quad z \to \infty. \tag{12}$$

Conversely, if the entire function $\Phi^*(z)$ satisfies the condition (12), then the function $w^*(z)$ defined from the equation (11) is a regular solution of the equation (4) and satisfies the condition (6). Note that, every polynomial $\Phi^*(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, $a_j \in \mathbb{C}$, $j = 0, 1, 2, \ldots, n$, satisfies the condition (12). Indeed this follows from the equalities:

$$\begin{aligned} \lim_{z\to\infty} \Phi^*(z)\, z^{-N}\, e^{-Q(z)} &= \lim_{z\to\infty} (a_0 z^n + a_1 z^{n-1} + \cdots + a_n)\, z^{-N}\, e^{-Q(z)} \\ &= a_0 \lim_{z\to\infty} z^{n-N}\, e^{-Q(z)} + a_1 \lim_{z\to\infty} z^{n-1-N}\, e^{-Q(z)} + \ldots \\ &+ a_n \lim_{z\to\infty} z^{-N}\, e^{-Q(z)} = 0. \end{aligned}$$

Here we used the fact, that $-Q(z)$ is the $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $-A(z)$, therefore

$$\begin{aligned} &\lim_{z\to\infty} z^{n-N}\, e^{-Q(z)} = 0, \quad \lim_{z\to\infty} z^{n-1-N}\, e^{-Q(z)} = 0, \ldots \\ &\lim_{z\to\infty} z^{-N}\, e^{-Q(z)} = 0. \end{aligned}$$

Hence every generalized polynomial corresponding to the equation (4) satisfies the condition (6). Since the space of all generalized polynomials corresponding to the equation (4) is infinite dimensional, the space of solutions of the equation (4) satisfying the condition (6) is infinite dimensional.

Let $\{w_j^*\}$, $j = 1, 2, 3, \ldots$, be an infinite system of linearly independent functions in this space. Let us prove that

$$w_j(z) = e^{-Q(z)}\, w_j^*(z), \quad j = 1, 2, 3, \ldots$$

is the infinite system of linearly independent functions in the space $\Omega(N)$. Since $w_j^*(z)$ is a regular solution of the equation (4) and satisfies the condition (6), it follows that the function $w(z)$ is a regular solution of the equation (1) satisfies the condition $w_j(z) = O(z^N)$ i.e. $w_j(z) \in \Omega(N)$.

Let $\sum_{j=1}^n c_j\, w_j(z) = 0, \quad z \in E, \quad c_j \in R, \quad j = 1, 2, \ldots, n$. Then we have

$$\sum_{j=1}^{n} c_j\, e^{-Q(z)}\, w_j^*(z) = 0, \quad z \in \mathbb{C}.$$

It follows from this equality, that $\sum_{j=1}^n c_j\, w_j^*(z) = 0$. Since the system of functions $\{w_j^*(z)\}$, $j = 1, 2, \ldots, n$, is linearly independent, one has $c_j = 0$, $j = 1, 2, \ldots, n$, i.e. we obtain, that the system of functions $\{w_j(z)\}$, $j = 1, 2, 3, \ldots$, is linearly independent. Consequently, the space $\Omega(N)$ is infinite dimensional.

Acknowledgements. The authors are grateful to the participants of the seminar "Elliptic systems on Riemann surfaces" at the Tbilisi State University for many discussions.

This paper was partially supported by the Rustaveli Foundation, Grant No 1-3/85.

R E F E R E N C E S

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Received 16.07.2009; revised 17.09.2009; accepted 10.11.2009.

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