Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 23, 2009

ON APPROXIMATE SOLUTION OF ONE NONLINEAR TWO-DIMENSIONAL DIFFUSION SYSTEM

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Abstract. The two-dimensional nonlinear system of partial differential equations arising in process of vein formation of young leaves is considered. Variable directions finite difference scheme is studied.

Keywords and phrases: System of nonlinear partial differential equations; variable directions finite difference schemes.

AMS subject classification: 35Q80; 65N06; 65Y99.

Consider the two-dimensional diffusion system of the nonlinear partial differential equations:

$$\begin{aligned}\frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left(V_1\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(V_2\frac{\partial U}{\partial y}\right),\\ \frac{\partial V_1}{\partial t} &= -V_1+g_1\left(V_1\frac{\partial U}{\partial x}\right),\\ \frac{\partial V_2}{\partial t} &= -V_2+g_2\left(V_2\frac{\partial U}{\partial y}\right),\end{aligned}\tag{1}$$

where g_α are given sufficiently smooth functions and following conditions are satisfied:

$$0<\gamma_0\le g_\alpha(\xi_\alpha)\le G_0,\quad \alpha=1,\,2,\quad \gamma_0=Const,\quad G_0=Const.$$

The model (1) describes the vein-formation in meristematic tissues of young leaves [1], U is the signal concentration and V_1, V_2 are diffusion coefficients for flux parallel to Ox and Oy axes, respectively.

The main features of systems (1) is that it contains equations of different order, which are strongly connected to each other. This fact dictates the usage of respective methods of research, because general theory is incompletely developed for such systems even in linear case. Naturally arises the questions of approximation of such problem, which also are connected with serious complexities as well.

In [1] some qualitative and structural properties of the system (1) are established. In [2] mathematical investigations are mainly carried out for one-dimensional analogue of system with one-component diffusion coefficient.

In many works the difference schemes belonging to the class of algorithms of splitting and variable directions for multidimensional models are constructed and studied (see, for example, [3]-[6] and references therein).

The averaged model of sum approximation as well as some discrete models for system (1) and its one-dimensional and multidimensional analogous are studied in [7-11].

In [7, 9-11] the construction and investigation of the splitting type and variable directions type schemes for system (1) and its multidimensional analogous are discussed for the initial-boundary value problem with Dirichlet boundary conditions.

In the present work we study the convergence of the scheme of the type of variable directions for the system (1) in the domain $[0,1]\times[0,1]\times[0,T]$ with the following boundary and initial data:

$$
\begin{gathered}
V_1(x,y,t)\frac{\partial U(x,y,t)}{\partial x}\bigg|_{x=0}=\mu_1(y,t),\quad V_2(x,y,t)\frac{\partial U(x,y,t)}{\partial y}\bigg|_{y=0}=\mu_2(x,t),\\
V_1(x,y,t)\frac{\partial U(x,y,t)}{\partial x}\bigg|_{x=1}=\eta_1(y,t),\quad V_2(x,y,t)\frac{\partial U(x,y,t)}{\partial y}\bigg|_{y=1}=\eta_2(x,t),\\
U(x,y,0)=U_0(x,y),\quad V_1(x,y,0)=V_{1,0}(x,y),\quad V_2(x,y,0)=V_{2,0}(x,y),
\end{gathered}
\tag{2}
$$

where μ_1, μ_2, η_1, η_2, U_0, $V_{1,0}$, $V_{2,0}$ are given sufficiently smooth functions. Let us note that boundary conditions here are dictated by biological viewpoint [1].

Using usual notations [5], let us correspond to the problem (1), (2) the difference scheme of the type of variable directions:

$$
\begin{gathered}
u_{1t}=(\hat v_1\hat u_{1\bar x})_x+(v_2u_{2\bar y})_y,\\
v_{1t}=-\hat v_1+g_1(v_1u_{1\bar x}),\\
v_1(0,y,t)u_{1x}(0,y,t)=\mu_1(y,t),\\
v_1(1,y,t)u_{1\bar x}(1,y,t)=\eta_1(y,t),\\
v_1(x,y,0)=V_{1,0}(x,y),
\end{gathered}
\tag{3}
$$

$$
\begin{gathered}
u_{2t}=(\hat v_1\hat u_{1\bar x})_x+(\hat v_2\hat u_{2\bar y})_y,\\
v_{2t}=-\hat v_2+g_2(v_2u_{2\bar y}),\\
v_2(x,0,t)u_{2y}(x,0,t)=\mu_2(x,t),\\
v_2(x,1,t)u_{2\bar y}(x,1,t)=\eta_2(x,t),\\
v_2(x,y,0)=V_{2,0}(x,y).
\end{gathered}
\tag{4}
$$

It is clear that each of the difference equations (3),(4) approximate the corresponding differential equations (1) and the conditions (2) under the sufficient smoothness of exact solution of the problem (1), (2) with the rate $O(\tau+h)$. Here τ and h are the steps with time and space variables, respectively.

Let us note that difference scheme of (3),(4) type for the problem (1),(2) with Dirichlet boundary conditions is constructed and studied in the work [11].

The following statement takes place.

Theorem. *If the initial-boundary value problem (1), (2) has the sufficiently smooth solution U, V_1, V_2, then the solution of the scheme of variable directions (3),(4) converges to the exact solution of problem (1), (2) when $\tau\to0$, $h\to0$ and for errors $Z_1=U-u_1$, $Z_2=U-u_2$, $S_1=V_1-v_1$, $S_2=V_2-v_2$ the following inequality holds*

$$
\|Z_{1\bar x}\|_1+\|Z_{2\bar y}\|_2+\|S_{1\bar x}\|_1+\|S_{2\bar y}\|_2\le C(\tau+h).
$$

Note that C in this theorem is a positive constant independent of τ and h.

The difference scheme (3),(4) is economical. It should be notice also that the algorithm does not apply iterative process as $\hat{v}_1$, $\hat{v}_2$ we find by explicit schemes and then $\hat{u}_1$, $\hat{u}_2$ - by solving resulting systems of linear algebraic equations with three-diagonal matrixes.

At last note that some other difference schemes can be also studied for the problem (1), (2). Particularly, following implicit difference scheme:

$$u_t = (\hat{v}_1 \hat{u}_{\bar{x}})_x + (\hat{v}_2 \hat{u}_{\bar{y}})_y,$$

$$v_{1t} = -\hat{v}_1 + g_1(v_1 u_{\bar{x}_1}), \quad v_{2t} = -\hat{v}_2 + g_2(v_2 u_{\bar{x}_2})$$

with suitable initial and boundary conditions corresponding to (2).

Many numerical experiments are carried out by using proposed algorithms. The results agree with theoretical conclusions. Article regarding results of numerical experiments and to their analysis will be published separately.

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Received 30.04.2009; accepted 10.09.2009.

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