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UNIQUENESS OF SOLUTIONS TO EXTERIOR BOUNDARY VALUE PROBLEMS OF THERMOELASTOSTATICS

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Abstract. We consider the Dirichlet and Neumann type exterior boundary value problems of thermoelasticty in the space of vector functions which are bounded at infinity and establish the structure of such solutions. On the basis of the results obtained we derive sufficient conditions which guarantee uniqueness of solutions.

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We investigate the uniqueness of solutions to the Dirichlet and Neumann type exterior boundary value problems (BVP) of thermoelasticty in the space of vector functions which are bounded at infinity. This kind of function space appears naturally in the study of nonhomogeneous BVPs of thermoelasticity for unbounded domains.

Let Ω^+ be a bounded domain of $\mathbb{R}^3$ with the boundary $S = \partial\Omega^+$. For simplicity, we assume that S is a $C^{2,\alpha}$-smooth surface with $0 < \alpha \leq 1$. We denote $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$.

The differential equations of thermoelastostatics read as ([1-3]),

$$\begin{aligned} &A(\partial)u(x) - \gamma \operatorname{grad} \vartheta(x) = \Phi(x), \\ &\Delta\vartheta(x) = \Phi_4(x), \end{aligned} \tag{1}$$

where $A(\partial)$ is the matrix differential operator generated by the classical Lamé equations,

$$A(\partial)u = \mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u,$$

λ and μ are the Lamé constants, γ ia thermal constant, $u = (u_1, u_2, u_3)$ is the displacement vector, $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ and $\Phi_4 - \Omega^-$ are given smooth functions with compact support.

For a surface element with the unit normal vector $n = (n_1, n_2, n_3)$, the stress vector in the thermoelasticity theory is calculated by formula

$$P(\partial, n)U = T(\partial, n)u - \gamma\, n\, \vartheta,$$

where $U = (u, \vartheta) = (u_1, u_2, u_3, \vartheta)^\top$, and

$$T(\partial, n)u = 2\mu \frac{\partial u}{\partial n} + \lambda n \operatorname{div} u + \mu[n \times \operatorname{rot} u].$$

Note that $T(\partial, n)u$ is the stress vector of the classical elasticity theory. Now we formulate the basic exterior BVPs.

Problem (D). Find a vector function $U \in [C^2(\Omega^-)]^4 \cap [C^1(\overline{\Omega^-})]^4$ which solves the system of differential equations (1) and satisfies the Dirichlet boundary condition on S

$$[\,u\,]^- = f, \qquad [\vartheta]^- = f_4.$$

Problem (N). Find a vector function $U \in [C^2(\Omega^-)]^4 \cap [C^1(\overline{\Omega^-})]^4$ which solves the system of differential equations (1) and satisfies the Neumann boundary condition on S

$$[\,P(\partial, n)U\,]^- = g, \qquad \Big[\frac{\partial \vartheta}{\partial n}\Big]^- = g_4.$$

Here $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$, f_4 and g_4 are given smooth functions on S.

It is clear that the boundary value problems for the temperature function ϑ is separated and we obtain the classical Dirichlet and Neumann problems for Poisson's equation, respectively: ϑ solves the following differential equation in Ω^-

$$\Delta\vartheta = \Phi_4$$

and on S satisfies either the Dirichlet condition

$$[\,\vartheta\,]^- = f_4$$

or the Neumann condition

$$\Big[\frac{\partial \vartheta}{\partial n}\Big]^- = g_4.$$

It is well known that in the space of functions which decay at infinity, the above BVPs for ϑ are uniquely solvable. Moreover, if $\vartheta = o(1)$ as $|x| \to \infty$, then

$$\vartheta(x) = \frac{\theta_0}{|x|} + \mathcal{O}(|x|^{-2}), \qquad \operatorname{grad}\vartheta(x) = -\frac{\theta_0}{|x|^3}\,x + \mathcal{O}(|x|^{-3}), \tag{2}$$

where

$$\theta_0 = -\frac{1}{4\pi}\int\limits_S \Big[\frac{\partial\vartheta(y)}{\partial n(y)}\Big]^- dS_y - \frac{1}{4\pi}\int\limits_{\Omega^*} \Phi_4(y)\,dy.$$

Here the domain $\Omega^* := \operatorname{supp}\Phi_4$ has a finite diameter.

Assuming that ϑ as a solution of the separated problem is a known function, form the above formulated basic boundary value problems we get the following BVPs for the nonhomogeneous Lamé equations.

Problem $(I)^-$. Find a solution vector $u \in [C^2(\Omega^-)]^3 \cap [C^1(\overline{\Omega^-})]^3$ which solves the system of differential equations in Ω^-

$$A(\partial)u = \Phi + \gamma\operatorname{grad}\vartheta, \tag{3}$$

and satisfies the Dirichlet boundary condition on S

$$[u]^- = f.$$

Problem $(II)^-$. Find a solution vector $u \in [C^2(\Omega^-)]^3 \cap [C^1(\overline{\Omega^-})]^3$ which solves the nonhomogeneous system of differential equations (3) in Ω^- and satisfies the Neumann boundary condition on S

$$T(\partial, n)u = g + \gamma\, n\,\vartheta.$$

From (2) it is easy to see that the right hand side vector function in (3) decays at infinity as $\mathcal{O}(|x|^{-2})$, in general. Therefore we have to look for solutions of the equation

(3) in the space of vector functions which are bounded at infinity. This complicates the investigation of BVPs under consideration.

First we prove some auxiliary assertions which are crucial in our analysis.

We rewrite equations (3) in the form

$$A(\partial)u(x) = -\frac{\theta_0\,\gamma}{|x|^3}\,x + \Psi(x) + \Phi(x), \quad x \in \Omega^-, \tag{4}$$

where

$$\Psi(x) = \gamma\Big[\ \mathrm{grad}\,\vartheta(x) + \frac{\theta_0}{|x|^3}\,x\Big].$$

It is clear that $\Psi(x) = \mathcal{O}(|x|^{-3})$ as $|x| \to \infty$. Recall that Φ has a compact support.

Lemma 1. *The vector*

$$u^{(0)}(x) = \vartheta_0\,\alpha\,\frac{x}{|x|} \tag{5}$$

with $\alpha = \frac{\gamma}{2(\lambda+2\mu)}$, *is a particular solution of the nonhomogeneous differential equation*

$$A(\partial)u = -\frac{\theta_0\,\gamma}{|x|^3}\,x, \quad x \in \mathbb{R}^3\backslash\{0\}.$$

Denote by $\Gamma(x-y)$ the fundamental matrix of the operator $A(\partial)$ ([3])

$$\Gamma(x) = [\,\Gamma_{kj}(x)\,]_{3\times 3}, \qquad \Gamma_{kj}(x) = \frac{\lambda'\delta_{kj}}{|x|} + \frac{\mu' x_k x_j}{|x|^3},$$

where δ_{kj} is the Kronecker symbol and

$$\lambda' = -\frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}, \quad \mu' = -\frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}.$$

Lemma 2. *Let* Ψ *and* Φ *be as in (4) and*

$$u^{(1)}(x) = \int\limits_{\Omega^-} \Gamma(x-y)\,[\,\Psi(y) + \Phi(y)\,]\,dy, \quad x \in \Omega^-, \tag{6}$$

Then $u^{(1)}$ *solves the differential equation*

$$A(\partial)u^{(1)}(x) = \Psi(x) + \Phi(x), \quad x \in \Omega^-,$$

and has the following asymptotic behaviour at infinity

$$u^{(1)}(x) = \mathcal{O}(|x|^{-1}\ln|x|) \quad \text{as} \quad |x| \to \infty.$$

Proof. Since Φ is smooth and has a compact support, we get

$$\int\limits_{\Omega^-} \Gamma(x-y)\,\Phi(y)\,dy = \mathcal{O}(|x|^{-1}) \quad \text{as} \quad |x| \to \infty.$$

On the other hand, since Ψ is smooth and $\Psi = \mathcal{O}(|x|^{-3})$ as $|x| \to \infty$, one can show by the appropriate decomposition of the domain of integration Ω^- that

$$\int_{\Omega^-} \Gamma(x-y)\Psi(y)dy = O(|x|^{-1}\ln|x|) \quad \text{as} \quad |x| \to \infty.$$

This proves the lemma.

Lemma 3. *Any bounded solution of the homogeneous equation*

$$A(\partial)u(x) = 0, \quad x \in \Omega^-, \tag{7}$$

can be represented at infinity in the form

$$u(x) = C + O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty,$$

where $C = (C_1, C_2, C_3)$ *is a constant vector.*

Proof. Let u be a solution of the homogeneous equation (7). Since $A(\partial)$ is an elliptic operator with constant coefficients, we have $u \in [C^\infty(\Omega^-)]^3$. Let $B(0,R)$ be a ball centered at the origin and radius $R > 0$ such that $\Omega^+ \subset B(0,R)$. Further, let w be a $C^\infty(\mathbb{R}^3)$-regular vector function with property:

$$w(x) = u(x) \ \text{ for } \ x \in \Omega^- \setminus B(0,R),$$

i.e., w is a extension of the vector function u from $\Omega^- \setminus B(0,R)$ onto the whole space $\mathbb{R}^3$. Then it is clear that

$$A(\partial)w(x) = G(x), \quad x \in \mathbb{R}^3, \tag{8}$$

where G is a vector function with compact support and $\operatorname{supp} G \subset \overline{B(0,R)}$.

Applying the generalized Fourier transform to equation (8) we get

$$A(-i\xi)\widehat{w}(\xi) = \widehat{G}(\xi), \quad \xi \in \mathbb{R}^3. \tag{9}$$

This equality is understood in the sense of the Schwartz space of tempered distributions $\mathcal{S}(\mathbb{R}^3)$. Since $\det A(-i\xi) = \mu^2\,(\lambda + 2\,\mu)\,|\xi|^6 \neq 0$ for $\xi \neq 0$, we conclude form (9)

$$\widehat{w}(\xi) = [\,A(-i\xi)\,]^{-1}\,\widehat{G}(\xi) + \sum_{|\beta|=0}^{M} C_\beta\,\delta^\beta(\xi),$$

where $\beta = (\beta_1, \beta_2, \beta_3)$ is a multi-index, $C_\beta = (C_{1\beta}, C_{2\beta}, C_{3\beta})$ are constant vectors, M is a natural number and $\delta(\cdot)$ is the Dirac's distribution. Therefore, with the help of the generalized inverse Fourier transform we arrive at the relation

$$\begin{aligned} w(x) &= \mathcal{F}^{-1}_{\xi \to x}[\,A^{-1}(-i\xi)\,\widehat{G}(\xi)\,] + \sum_{|\beta|=0}^{M} C_\beta\,x^\beta \\ &= \int_{\mathbb{R}^3} \Gamma(x-y)\,G(y)\,dy + \sum_{|\beta|=0}^{M} C_\beta\,x^\beta. \end{aligned} \tag{10}$$

Here we applied that $\mathcal{F}^{-1}_{\xi\to x}[\,A^{-1}(-i\xi)] = \Gamma(x)$ (see, e.g., [1]).

Now, taking into account that G has a compact support, we conclude that the first summand in the right hand side in (10) decays at infinity as $O(|x|^{-1})$ as $|x| \to \infty$. Due to the boundedness of w at infinity, it follows that $C_\beta = 0$ for all β with $|\beta| \neq 0$. Therefore we finally get

$$w(x) = \int\limits_{\mathbb{R}^3} \Gamma(x-y)\, G(y)\, dy + C_0,$$

with a constant vector C_0. Since the support of G is compact the proof follows.

Lemma 4. *Any solution of equation (6) is representable in the form*

$$u(x) = u^{(0)}(x) + u^{(1)}(x) + v(x) + C, \tag{11}$$

where $u^{(0)}$ and $u^{(1)}$ are given by formulas (5) and (6) respectively, $C = (C_1, C_2, C_3)$ is a constant vector, and v is a solution to the homogeneous equation $A(\partial)v(x) = 0$ in Ω^- which decays at infinity as $v(x) = \mathcal{O}(|x|^{-1})$.

If u satisfies the condition

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma(0,R)} u(x)\, d\Sigma(0,R) = 0, \tag{12}$$

where $\Sigma(0,R) := \partial B(0,R)$, then the constant vector C in (11) vanishes.

Proof. It follows immediately from Lemmas 1-3.

Denote by $H(\Omega^-)$ the class of vector functions $U = (u, \vartheta) \in [C^2(\Omega^-)]^4 \cap [C^1(\overline{\Omega^-})]^4$, such that $\vartheta(x) = o(1)$ and $u(x) = \mathcal{O}(1)$ as $|x| \to \infty$ and, in addition, u satisfies the condition (12).

Now we are in the position to prove the following basic theorem.

Theorem 5. *The exterior boundary value problems (D) and (N) have at most one solution in the class of vector functions $H(\Omega^-)$.*

Proof. Due to the linearity of the problems in question, it suffices to prove that the homogeneous BVPs (D) and (N) have only the trivial solution. It is clear that $\vartheta(x) = 0$ in Ω^-, since the homogeneous exterior Dirichlet and Neumann BVPs for the Laplace equation possess only the trivial solutions in the space of functions decaying at infinity. Therefore the displacement vector u solves the homogenous equation $A(\partial)u = 0$ in Ω^-, satisfies either the homogeneous Dirichlet type condition $[u]^- = 0$ or the Neumann type condition $[Tu]^- = 0$ on S. Moreover, u decays at infinity as

$$u = O(|x|^{-1} \ln |x|), \quad |x| \to \infty,$$

due to the Lemmas 1-4 and Theorem 5. One can easily show that for such vector functions there holds Green's identity

$$\int\limits_{\Omega^-} A(\partial)u \cdot u\, dx = -\int\limits_{S} [\,Tu\,]^- \cdot [\,u\,]^-\, dS + \int\limits_{\Omega^-} E(u,u)\, dx,$$

where the so called potential energy density $E(u,u)$ is a positive definite quadratic form with respect to the deformations $e_{kj}=2^{-1}(\partial_j u_k+\partial_k u_j)$,

$$E(u,u)\geq \delta_0\sum_{k,j=1}^{3}e_{kj}^2,\quad \delta_0=const>0.$$

Due to the homogeneity of the BVPs, from the above Green's formula it follows that $E(u,u)=0$ in Ω^-. Therefore u is a rigid displacement vector $u(x)=[a\times x]+b$ in Ω^-, and since $u\in H(\Omega^-)$ finally we conclude that $u(x)=0$ in Ω^-.

R E F E R E N C E S

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