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THE STRESSES CONCENTRATION PROBLEM FOR CYLINDRICAL SHELLS ON THE I. VEKUA'S HIGH APPROXIMATIONS

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Abstract. In the present paper on the basis of I. Vekua' s theory (approximate $N = 0, 1, 2$) we consider well-known problem of stresses concentration for shallow and non-shallow cylindrical shell. To solve the problems algorithm of full automation is devised by means of the net method. The program named VEKMUS is constructed [2]. By means of the program the problems of stresses concentration shallow and non-shallow cylindrical shells are solved for the approximations $N = 0, 1, 2$.

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I. Vekua's complete system of 2-D equations of non-shallow cylindrical shells for the any approximation of order N have the form [1]:

a) equations of equilibrium

$$\begin{cases} \nabla_\alpha \overset{(m)}{\sigma_1^\alpha} - \dfrac{2m+1}{h}\Big(\overset{(m-1)}{\sigma_1^3} + \overset{(m-3)}{\sigma_1^3} + \cdots\Big) + \overset{(m)}{F_1} = 0, \\ \nabla_\alpha \overset{(m)}{\sigma_2^\alpha} + \dfrac{1}{R}\overset{(m)}{\sigma_2^3} - \dfrac{2m+1}{h}\Big(\overset{(m-1)}{\sigma_2^3} + \overset{(m-3)}{\sigma_2^3} + \cdots\Big) + \overset{(m)}{F_2} = 0, \\ \nabla_\alpha \overset{(m)}{\sigma_3^\alpha} - \dfrac{1}{R}\overset{(m)}{\sigma_2^3} - \dfrac{2m+1}{h}\Big(\overset{(m-1)}{\sigma_3^3} + \overset{(m-3)}{\sigma_3^3} + \cdots\Big) + \overset{(m)}{F_3} = 0, \end{cases} \tag{1}$$

where

$$\Big(\overset{(m)}{\sigma_j^i}, \overset{(m)}{\phi_j}\Big) = \frac{2m+1}{2h}\int_{-h}^{h}\left(\sqrt{\frac{g}{a}}\,\sigma_j^i, \sqrt{\frac{g}{a}}\,\phi_j\right) P_m\left(\frac{x_3}{h}\right)dx_3, \quad (i,j=1,2,3),$$

$$\overset{(m)}{F_i} = \overset{(m)}{\phi_i} + \frac{2m+1}{2h}\left[\sqrt{\frac{g_+}{a}}\,\overset{(+)}{\sigma_i^3} - (-1)^m\sqrt{\frac{g_-}{a}}\,\overset{(-)}{\sigma_i^3}\right], \quad \sqrt{\frac{g_\pm}{a}} = 1 \pm \frac{h}{R},$$

$$\overset{(\pm)}{\sigma^3} = \sigma^3(x^1, x^2, \pm h), \quad (m = 0, 1, ..., N),$$

$$-h \le x_3 = x^3 \le h.$$

b) For the Hooke's law we have [3]

$$\overset{(m)}{\sigma_{11}} = (\lambda + 2\mu)\left[\partial_1\overset{(m)}{u_1} + \frac{h}{R}\partial_1\underline{\overset{(m)}{u_1}}\right] + \lambda\left[\left(\partial_2\overset{(m)}{u_2} + \frac{1}{R}\overset{(m)}{u_3}\right) + \overset{(m)}{u_3'} + \frac{h}{R}\overset{(m)}{u_3''}\right],$$

$$\overset{(m)}{\sigma_{12}} = \mu\left[\partial_2\overset{(m)}{u_1} + \partial_1\overset{(m)}{u_2} + \frac{h}{R}\partial_1\underline{\overset{(m)}{u_2}}\right], \qquad \overset{(m)}{\sigma_{21}} = \mu\left[\partial_1\overset{(m)}{u_2} + \sum_{s=0}^{N} A_{ms}\partial_2\overset{(s)}{u_1}\right],$$

$$\overset{(m)}{\sigma}_{22} = (\lambda+2\mu)\sum_{s=0}^{N} A_{ms}\Big(\partial_2 \overset{(s)}{u}_2 + \frac{1}{R}\overset{(s)}{u}_3\Big) + \lambda\Big(\partial_1 \overset{(m)}{u}_1 + \overset{(m)}{u'_3}\Big), \tag{2}$$

$$\overset{(m)}{\sigma}_{13} = \overset{(m)}{\sigma}_{31} = \mu\Big[\partial_1 \overset{(m)}{u}_3 + \overset{(m)}{u'_1} + \frac{h}{R}\partial_1 \overset{(m)}{\underline{u_2}}\Big], \quad \overset{(m)}{\sigma}_{23} = \mu\Big[\overset{(m)}{u'_2} + \sum_{s=0}^{N} A_{ms}\Big(\partial_2 \overset{(m)}{u}_3 - \frac{1}{R}\overset{(s)}{u}_3\Big)\Big],$$

$$\overset{(m)}{\sigma}_{32} = \mu\Big[\overset{(m)}{u'_2} + \frac{h}{R}\overset{(m)}{u''_2} + \partial_2 \overset{(m)}{u}_3 - \frac{1}{R}\overset{(m)}{u}_2\Big],$$

$$\overset{(m)}{\sigma}_{33} = \mu\Big[\partial_1 \overset{(m)}{u}_1 + \partial_2 \overset{(m)}{u}_2 + \frac{1}{R}\overset{(m)}{u}_3 + \frac{h}{R}\partial_1 \overset{(m)}{\underline{u_1}}\Big] + (\lambda+2\mu)\Big(\overset{(m)}{u'_3} + \frac{h}{R}\overset{(m)}{u''_3}\Big),$$

where

$$\overset{(m)}{u_i} = \frac{2m+1}{h}\int_{-h}^{h} u_i\, P_m\Big(\frac{x_3}{h}\Big)dx_3, \qquad \overset{(m)}{u'_i} = \frac{2m+1}{h}\Big(\overset{(m+1)}{u_i} + \overset{(m+3)}{u_i} + \cdots\Big),$$

$$\overset{(m)}{\underline{u_i}} = \frac{m}{2m-1}\overset{(m-1)}{u_i} + \frac{m+1}{2m+3}\overset{(m+1)}{u_i}, \quad \overset{(m)}{u''_i} = \frac{1}{h}\Big[m\overset{(m)}{u_i} + (2m+1)\Big(\overset{(m+2)}{u_i} + \overset{(m+4)}{u_i} + \cdots\Big)\Big],$$

$$A_{ms} = \frac{2m+1}{2h}\int_{-h}^{h} \frac{P_m\big(\frac{x_3}{h}\big)\,P_s\big(\frac{x_s}{h}\big)dx_3}{1+\frac{x_3}{R}}$$

$$= (-1)^{m+s}(2m+1)\,\frac{R}{h}\left[\begin{array}{ll} P_m\Big(\dfrac{R}{h}\Big)Q_s\Big(\dfrac{R}{h}\Big), & m \le s, \\ Q_m\Big(\dfrac{R}{h}\Big)P_s\Big(\dfrac{R}{h}\Big), & m \ge s. \end{array}\right.$$

Here $Q_s(x)$ is the Legendre function of second order.

For shallow cylindrical shells $\partial_\alpha \overset{(m)}{\underline{u^i}} = 0$ and $\overset{(m)}{u''_i} = 0$.

On bases I. Vekua's approximate $N = 0, 1, 2$ an automatic numerical program named VEKMUS is compiled to calculate the stress concentration for the cylindrical shells weakened by rectangular holes.

Let $u_i = u_i(x_1, x_2)$ are two-differentiable functions on some domain ω. $A_{klm} = A_{klm}(x_1, x_2)$, $f_k = f_k(x_1, x_2)$ are given continuous functions on the same domain ω, $k, l = \overline{1, n}$, $m = \overline{1, 6}$. In the program VEKMUS the algorithm of interchanging the following system equations by the finite-difference scheme is constructed:

$$A_{k11}\frac{\partial^2 u_1}{\partial x_1^2} + A_{k12}\frac{\partial^2 u_1}{\partial x_1\,\partial x_2} + A_{k13}\frac{\partial^2 u_1}{\partial x_2^2} + A_{k14}\frac{\partial u_1}{\partial x_1} + A_{k15}\frac{\partial u_1}{\partial x_2} + A_{k16}u_1$$
$$+ A_{k21}\frac{\partial^2 u_2}{\partial x_1^2} + A_{k22}\frac{\partial^2 u_2}{\partial x_1\,\partial x_2} + A_{k23}\frac{\partial^2 u_2}{\partial x_2^2} + A_{k24}\frac{\partial u_2}{\partial x_1} + A_{k25}\frac{\partial u_2}{\partial x_2} + A_{k26}u_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ A_{kn1}\frac{\partial^2 u_n}{\partial x_1^2} + A_{kn2}\frac{\partial^2 u_n}{\partial x_1\,\partial x_2} + A_{kn3}\frac{\partial^2 u_n}{\partial x_2^2} + A_{kn4}\frac{\partial u_n}{\partial x_1} + A_{kn5}\frac{\partial u_n}{\partial x_2} + A_{kn6}u_n = f_k. \tag{3}$$

It is easy to see that of Vekua's shell theory or plane elasticity theory are particular cases of the system (3). By this reason above mentioned algorithm will be served equally by this two theories. The corresponding to (3) finite-difference scheme will be constructed automatically in the matrix-vector form

$$\begin{cases} B_0W_0 + C_0W_1 = F_1, \\ A_1W_0 + B_1W_1 + C_1W_2 = F_2, \\ \qquad A_2W_1 + B_2W_2 + C_2W_3 = F_3, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \qquad\qquad A_{M-1}W_{M-2}+B_{M-1}W_{M-1}+C_{M-1}W_M = F_{M-1}, \\ \qquad\qquad\qquad A_MW_{M-1} + B_MW_M = F_M, \end{cases} \tag{4}$$

where A_i, B_i, C_i, $i = \overline{0, M}$, are the quadratic matrix of order nN of the same structure,

$$F_i = \big[(f_1, f_2, \dots, f_n)_{i0}, (f_1, f_2, \dots, f_n)_{i1}, \dots, (f_1, f_2, \dots, f_n)_{iN}\big],$$
$$W_i = \big[(u_1, u_2, \dots, u_n)_{i0}, (u_1, u_2, \dots, u_n)_{i1}, \dots, (u_1, u_2, \dots, u_n)_{iN}\big],$$

are the vectors of order nN, n is the number of unknowns on nodal points of the net, N and M are the discretization parameters of the net.

From the system (4) we get easily the algorithm of its solution

$$\begin{aligned} &W_0 = B_0^{-1}(F_0 - C_0\,W_1), \quad W_1 = X_1\,W_2 + Y_1, \\ &W_i = X_i\,W_{i+1} + Y_i, \qquad W_M = X_M\,Y_M, \quad (i = 2, 3, ..., M{-}1), \end{aligned} \tag{5}$$

where

$$\begin{aligned} &X_1 = -\big(B_1 - A_1\,B_0^{-1}\,C_0\big)^{-1}C_1, \quad Y_1 = \big(B_1 - A_1\,B_0^{-1}\,C_0\big)^{-1}\big(F_1 - A_1\,B_0^{-1}\,F_0\big), \\ &X_i = -\big(A_i\,X_{i-1} + B_i\big)^{-1}C_i, \qquad Y_i = \big(A_i\,X_{i-1} + B_i\big)^{-1}\big(F_i - A_i\,Y_{i-1}\big), \\ &X_M = -\big(A_M\,X_{M-1} + B_M\big)^{-1}, \quad Y_M = F_M - A_M\,Y_{M-1}, \quad (i = 2, 3, ..., M{-}1). \end{aligned} \tag{6}$$

The formulas (5) and (6) are algorithms of matrix factorization. On the first step the coefficients $X_1, Y_1, \dots, X_M, Y_M$ are calculated (the direct step), and on the second step the desired vectors $W_M, W_{M-1}, \dots, W_0$ are calculated (inverse step).

The stresses concentration problem. Let we have the cylindrical shell (Fig. 1), the lateral surfaces of which are loading by uniformly distributed tension force Q. Denote by P the intensity of this force. Other surfaces are free. Determine the stress state of the shell.

boundary curves	boundary conditions
AB, CD	$\overset{(0)}{\sigma}_{11} = p$, $\overset{(m)}{\sigma}_{ij} = 0,\ m = 0, 1, 2$
AD, BC, A_1D_1, B_1C_1, A_1B_1, D_1C_1	$\overset{(m)}{\sigma}_{ij} = 0,\ m = 0, 1, 2$

The formulated problem was solved by VEKMUS for shallow and non-shallow shells on the domain $\omega_h^{100}\big(h = \frac{1}{100}\big)$.

From the obtained results we give the short analysis of the solutions.

The concentration of stress are high valued at the neighbourhoods of the points A_1, B_1, C_1, D_1.

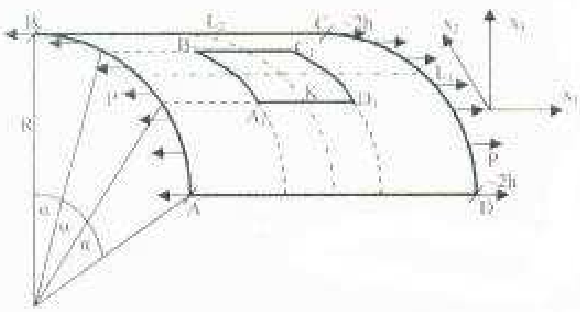

Fig.1.

	$max\|\sigma_{11}\|$	$max\|\sigma_{12}\|$	$max\|\sigma_{22}\|$	$max\|\sigma_{11}(k)\|$
N=0 shellow shells	$21p$	$17p$	$18p$	$1,5p$
N=0 nonshellow shells	$21p$	$17p$	$18p$	$1,5p$
N=1 shellow shells	$70p$	$63p$	$74p$	$4p$
N=1 nonshellow shells	$69p$	$62p$	$73p$	$4p$
N=2 shellow shells	$29p$	$27p$	$33p$	$3,9p$
N=2 nonshellow shells	$37p$	$34p$	$40p$	$2,5p$

$$E = 2,1 \cdot 10^6;\ \sigma = 0.3;\ \alpha = \pi/6;\ R = 200 \text{ sm};\ 2h = 4 \text{ sm};\ |AB| = |AD| = 200^*\pi/2 \text{ sm},$$
$$|A_1B_1| = |A_1D_1| = 200^*\pi/6sm, P = 1.$$

R E F E R E N C E S

1. Vekua I. Shell theory: General Methods of Construction. *Pitman advanced publishing program, Boston–London–Melbourne–New York*, 1985.

2. Gogia A. Computation of a cylindrical shell with holes. *Bull. Georgian acad. sci.* **115** (1984), 41-44.

3. Gogia A., Meunargia T. Some problem of the stresses concentration for non-shallow cylindrical shells on the basis of I. Vekua's theory. *AMIM,* **1** (2007), 39-51.

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