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ON THE DERIVABILITY AND REPRESENTATIONS OF QUATERNION FUNCTIONS

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Abstract. For the quaternion functions of a quaternion variable we introduce the notion of a $\mathbb{Q}$-derivative. In particular, it is proved that the elementary functions introduced by Hamilton possess such a derivative. The $\mathbb{Q}$-derivation rules are established, and the necessary and sufficient conditions are found for the existence of a $\mathbb{Q}$-derivative. The properties of quaternion functions are investigated with respect to two complex variables, and both their integral representation and their representation by power series are given.

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1. Introduction. The important theory of holomorphic (analytic) functions of one complex variable with comprehensive applications to various problems of natural sciences gave a serious impetus to the search for analogous theories for functions of three and more real variables.

It turned out that an analogous theory, following Frobenius' theorem [4], did not exist for functions of three real variables.

To find an analogous theory for functions of four real variables, in 1843 Hamilton introduced quaternions in the consideration. The quaternion units i_0, i_1, i_2 and i_3 introduced by Hamilton are subject to the conditions: $i_0 = 1$, $i_1^2 = i_2^2 = i_3^2 = -1$, $i_1 i_2 = -i_2 i_1 = i_3$, $i_2 i_3 = -i_3 i_2 = i_1$, $i_3 i_1 = -i_1 i_3 = i_2$. The quaternions $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ with norm $|x| = (x_0^2 + x_1^2 + x_2^2 + x_3^3)^{\frac{1}{2}}$ are assigned to each point (x_0, x_1, x_2, x_3) from the real four-dimensional space $\mathbb{R}^4$. The space $\mathbb{R}^4$ is thus identified with the body of quaternions $\mathbb{Q}$.

It is natural to construct the theory of quaternion functions $f(z) = u_0(z) + u_1(z)i_1 + u_2(z)i_2 + u_3(z)i_3$, where $u_k(z)$ are real functions, using the scheme by which the theory of holomorphic functions of one complex variable is constructed provided that such a scheme exists. There are three well known methods of construction of the theory of holomorphic functions of one complex variable. These methods are nor applicable to the quaternion functions.

Here for the quaternion functions of a quaternion variable we give the notion of a $\mathbb{Q}$-derivative which all elementary functions have. The rules of $\mathbb{Q}$-derivation are established and the necessary and sufficient conditions for the existence of a $\mathbb{Q}$-derivative are found similarly to the case of complex functions of one complex variable. The notions and conditions of $\mathbb{C}^2$-differentiability and $\mathbb{C}^2$-holomorphy for quaternion functions are given with respect to two complex variables. The integral representation and the representation by power series are discussed for $\mathbb{C}^2$-holomorphic functions.

2. The Notion of a $\mathbb{Q}$-Derivative

Definition 2.1. A quaternion function $f(z)$, $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, defined in the neighborhood of a point $z^0 = x_0^0 + x_1^0 i_1 + x_2^0 i_2 + x_3^0 i_3$ is called $\mathbb{Q}$-derivable at z^0 if there exist two numerical sequences of quaternions $A_k(z^0)$ and $B_k(z^0)$ such that $\sum\limits_k A_k(z^0)B_k(z^0)$ is finite and the equality

$$f(z^0 + h) - f(z^0) = \sum_k A_k(z^0) \cdot h \cdot B_k(z^0) + \omega(z^0, h)$$

is valid, where $\lim\limits_{h\to 0} \frac{|\omega(z^0,h)|}{|h|} = 0$. Moreover, the quaternion $\sum\limits_k A_k(z^0)B_k(z^0)$ is called the $\mathbb{Q}$-derivative of the function f at the point z^0 and we write

$$f'(z^0) = \sum_k A_k(z^0)B_k(z^0).$$

Theorem 2.2. *The following equalities are valid:*

$$(z^n)' = nz^{n-1}, \quad n = 0, 1, 2, \dots, \quad (e^z)' = e^z, \quad (\sin z)' = cosz, \quad (\cos z)' = -\sin z.$$

3. The $\mathbb{Q}$-Derivation Rule. The equalities $(c \cdot f(z))' = c \cdot f'(z)$ and $(f(x) \cdot c)' = f'(x) \cdot c$ are obvious when there exists a derivative $f'(z)$, where c is a constant quaternion.

Proposition 3.1. *If there exist $f'(z)$ and $\varphi'(z)$, then*

$$(f + \varphi)'(z) = f'(z) + \varphi'(z).$$

Proposition 3.2. *Let there exist $f'(z)$ and $\varphi'(z)$. Then*

$$(f \cdot \varphi)'(z) = f'(z)\varphi(z) + f(z)\varphi'(z).$$

Corollary 3.3. If $f_1, f_2, \dots, f_n$ are $\mathbb{Q}$-derivable functions at a point z, then

$$(f_1 \cdot f_2 \cdots f_n)'(z) = f_1' \cdot f_2 \cdots f_n + f_1 \cdot f_2' \cdot f_3 \cdots f_n + \cdots + f_1 \cdot f_2 \cdots f_{n-1} \cdot f_n'.$$

Corollary 3.4. If a function f is $\mathbb{Q}$-derivable at a point z, then for $n = 1, 2, \dots$

$$(f^n)' = f' \cdot f^{n-1} + f \cdot f' \cdot f^{n-2} + \cdots + f^{n-1} \cdot f'.$$

Proposition 3.5. *If there exist $\varphi'(z)$ and $\varphi \neq 0$ in a neighborhood of a point z, then at z there holds the equality*

$$\left(\frac{1}{\varphi}\right)'(z) = -\frac{1}{\varphi(z)} \cdot \varphi'(z) \cdot \frac{1}{\varphi(z)}.$$

Corollary 3.6. For $z \neq 0$ we have $(z^m)' = mz^{m-1}$, $m = -1, -2, \dots$, and $\left(\frac{1}{c-z}\right)' = \frac{1}{(c-z)^2}$ $(c \neq z)$.

Corollary 3.7. If there exist $f'(z)$, $\varphi'(z)$ and $\varphi \neq 0$ in a neighborhood of a point z, then

$$\left(f \cdot \frac{1}{\varphi}\right)'(z) = f'(z) \cdot \frac{1}{\varphi(z)} - f(z)\,\frac{1}{\varphi(z)} \cdot \varphi'(z) \cdot \frac{1}{\varphi(z)}\,,$$

$$\left(\frac{1}{\varphi} \cdot f\right)'(z) = -\frac{1}{\varphi(z)} \cdot \varphi'(z) \cdot \frac{1}{\varphi(z)} f(z) + \frac{1}{\varphi(z)} \cdot f'(z)\,.$$

Corollary 3.8. If there exists f', then

$$(f^n)' = f^{n-1} \cdot f' + f^{n-2} \cdot f' \cdot f + f^{n-3} \cdot f' \cdot f^2 + \cdots + f' \cdot f^{n-1},$$

$$(f^n)' = f' \cdot f^{n-1} + f \cdot f' \cdot f^{n-2} + f^2 \cdot f' \cdot f^{n-3} + \cdots + f^{n-1} \cdot f'.$$

4. The $\mathbb{Q}$-Derivative of the Logarithm. A quaternion w is called the logarithm of a finite quaternion $z \neq 0$ if the equality $z = e^w$ is fulfilled and we write $w = \ln z$.

To define the $\mathbb{Q}$-derivative $w' = (\ln z)'$, we have

$$1 = \left(1 + \frac{w}{2!} + \frac{w^2}{3!} + \cdots\right) \cdot w' + \left(\frac{1}{2!} + \frac{w}{3!} + \frac{w^2}{4!} + \cdots\right) \cdot w' \cdot w$$

$$+ \left(\frac{1}{3!} + \frac{w}{4!} + \frac{w^2}{5!} + \cdots\right) \cdot w' \cdot w^2 + \cdots\ .$$

5. Necessary and Sufficient Conditions for $\mathbb{Q}$-Derivability. Here we will give the assertions on the relations between the $\mathbb{Q}$-derivability and the differentiability of quaternion functions.

Theorem 5.1. *If a quaternion function F of a quaternion variable is $\mathbb{Q}$-derivable at a point z, then f is differentiable at z and its angular partial derivatives* [1] $f'_{\widehat{x}_0}$, $f'_{\widehat{x}_1}$, $f'_{\widehat{x}_2}$, $f'_{\widehat{x}_3}$ *are related to the derivative $f'(z) = \sum\limits_k A_k(z)B_k(z)$ through the equalities*

$$f'_{\widehat{x}_0}(z) = \sum_k A_k(z)B_k(z) = f'(z), \quad f'_{\widehat{x}_1}(z) = \sum_k A_k(z)i_1B_k(z), \tag{1}$$

$$f'_{\widehat{x}_2}(z) = \sum_k A_k(z)i_2B_k(z), \quad f'_{\widehat{x}_3}(z) = \sum_k A_k(z)i_3B_k(z). \tag{2}$$

Theorem 5.2. *Let a quaternion function f of a quaternion variable be differentiable at a point z. Assume that its finite angular partial derivatives $f'_{\widehat{x}_0}$, $f'_{\widehat{x}_1}$, $f'_{\widehat{x}_2}$, $f'_{\widehat{x}_3}$ existing at z admit representations* (1)–(2). *Then the function f has the $\mathbb{Q}$-derivative $f'(z)$ at the point z and $f'(z) = \sum_k A_k(z)B_k(z)$.*

Theorem 5.3. *If a function f is $\mathbb{Q}$-derivable at a point z and $f'(z) = \sum\limits_k A_k(z)B_k(z)$, then its differential $df(z) = f'_{\widehat{x}_0}(z)\,dz_0 + f'_{\widehat{x}_1}(z)\,dx_1 + f'_{\widehat{x}_2}(z)\,dx_2 + f'_{\widehat{x}_3}(z)\,dx_3$ admits the representation $df(z) = \sum_k A_k(z)\,dz\,B_k(z)$, where $dz = dx_0 + i_1dx_1 + i_2dx_2 + i_3dx_3$.*

6. The $\mathbb{C}^2$-Differentiability and $\mathbb{C}^2$-Holomorphy of Quaternion Functions

6A. $\mathbb{C}^2$-Differentiability of quaternion functions.

Definition 6.1. A quaternion function $f(z) = f_1(z) + f_2(z)\,i_2$, $z = (z_1, z_2) = z_1 + z_2 i_2$, is called $\mathbb{C}^2$-differentiable at the point $z^0 = (z_1^0, z_2^0) = z_1^0 + z_2^0 i_2$ if there exist quaternion numbers $d_1 + d_1' i_2$ and $d_2 + d_2' i_2$ such that the condition

$$\lim_{z\to z^0} \frac{f(z) - f(z^0) - \sum\limits_{k=1}^{2} (z_k - z_k^0)(d_k + d_k' i_2)}{\|z - z^0\|} = 0$$

is fulfilled.

In that case, the sum $\sum\limits_{k=1}^{2} (z_k - z_k^0)(d_k + d_k' i_2)$ is called the $\mathbb{C}^2$-differential of the quaternion function $f(z)$ at the point z^0.

Theorem 6.2. *For a quaternion function $f(z) = f_1(z)+f_2(z)i_2$ to be $\mathbb{C}^2$-differentiable at a point z^0 it is necessary and sufficient that one of the following conditions be fulfilled:*

(i) *the complex functions $f_1(z)$ and $f_2(z)$ are $\mathbb{C}^2$-differentiable at the point z^0;*

(ii) *the equalities*

$$\frac{\partial f}{\partial \widehat{x}_0}\,(z^0) + i_1\,\frac{\partial f}{\partial \widehat{x}_1}\,(z^0) = 0, \qquad \frac{\partial f}{\partial \widehat{x}_2}\,(z^0) + i_1\,\frac{\partial f}{\partial \widehat{x}_3}\,(z^0) = 0$$

are fulfilled;

(iii) *the equality*

$$df(z^0) = dz_1\,\frac{\partial f}{\partial \widehat{z}_1}\,(z^0) + dz_2\,\frac{\partial f}{\partial \widehat{z}_2}\,(z^0)$$

is true; here $\frac{\partial f}{\partial \widehat{z}_1} = \frac{\partial f_1}{\partial \widehat{z}_1} + \frac{\partial f_2}{\partial \widehat{z}_1}\,i_2$, $\frac{\partial f}{\partial \widehat{z}_2} = \frac{\partial f_1}{\partial \widehat{z}_2} + \frac{\partial f_2}{\partial \widehat{z}_2}\,i_2$ and for a complex function $g(z_1, z_2)$ of two complex variables the formal angular partial derivatives with respect to z_1 and z_2 are introduced by the equalities [2]

$$\frac{\partial g}{\partial \widehat{z}_1} = \frac{1}{2}\Big(\frac{\partial g}{\partial \widehat{x}_0} - i_1\,\frac{\partial g}{\partial \widehat{x}_1}\Big), \qquad \frac{\partial g}{\partial \widehat{z}_2} = \frac{1}{2}\Big(\frac{\partial g}{\partial \widehat{x}_2} - i_1\,\frac{\partial g}{\partial \widehat{x}_3}\Big).$$

6B. $\mathbb{C}^2$-Holomorphy of quaternion functions.

The conditions of Theorem 6.2 as to the $\mathbb{C}^2$-differentiability of a quaternion function allow us to introduce

Definition 6.3. A quaternion function $f(z) = f_1(z)+f_2(z)i_2$ is called $\mathbb{C}^2$-holomorphic at a point z^0 or in a domain $D \subset \mathbb{C}^2$ if f is $\mathbb{C}^2$-differentiable in a neighborhood of z^0 or at each point of the domain D.

Proposition 6.4. *For a quaternion function $f(z)$ to be $\mathbb{C}^2$-holomorphic at a point z^0 or in a domain $D \subset \mathbb{C}^2$ it is necessary and sufficient that conditions* (i)–(iii) *from Theorem 6.2 be fulfilled, each separately, in a neighborhood of the point z^0 or at each point of the domain D.*

Proposition 6.5. *The $\mathbb{C}^2$-holomorphy at a point or in a domain of the quaternion function $f(z) = f_1(z) + f_2(z)i_2$ is equivalent to the simultaneous $\mathbb{C}^2$-holomorphy at a point or in a domain of the complex functions $f_1(z)$ and $f_2(z)$.*

7. Integral Representations of $\mathbb{C}^2$-Holomorphic Quaternion Functions

Theorem 7.1. *Let a quaternion function $f(z) = f_1(z)+f_2(z)i_2$ be $\mathbb{C}^2$-holomorphic in a domain $D \subset \mathbb{C}^2$ which is the Cartesian product of the 1-connected domains $D_1 \subset \mathbb{C}$ and $D_2 \subset \mathbb{C}$. Then at every point $z = (z_1, z_2) \in D$ we have the integral representation*

$$f(z_1, z_2) = -\frac{1}{4\pi^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{d\zeta_1\, d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, f(\zeta_1, \zeta_2),$$

where Γ_1 and Γ_2 are any closed paths in D_1 and D_2, respectively, containing within themselves the points z_1 and z_2.

Theorem 7.2. *If a quaternion function $f(z_1, z_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2$ is $\mathbb{C}^2$-holomorphic in the Cartesian product of the* 1*-connected domains $D_1 \subset \mathbb{C}$ and $D_2 \subset \mathbb{C}$, then its partial derivatives $f'_{z_1}(z_1, z_2)$ and $f'_{z_2}(z_1, z_2)$ are also $\mathbb{C}^2$-holomorphic quaternion functions in $D_1 \times D_1 \subset \mathbb{C}^2$.*

8. Representations of $\mathbb{C}^2$-Holomorphic Quaternion Functions by Power Series

Theorem 8.1. *Let a quaternion function $f(z) = f_1(z)+f_2(z)i_2$ be $\mathbb{C}^2$-holomorphic in the domain $D \subset \mathbb{C}^2$ which is the Cartesian product of* 1*-connected domains $D_1 \subset \mathbb{C}$ and $D_2 \subset \mathbb{C}$. Then at every point $z^0 = (z_1^0, z_2^0) \in D$ from the neighborhood of a point $z^0 = (z_1^0, z_2^0) \in D$ we have the representation by a power series*

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} (z_1 - z_1^0)^m (z_2 - z_2^0)^n \cdot c_{mn}\,,$$

where the quaternion coefficients c_{mn} of the function f are given by the equalities

$$c_{mn} = -\frac{1}{4\pi^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{d\zeta_1\, d\zeta_2}{(\zeta_1 - z_1^0)^{m+1}(\zeta_2 - z_2^0)^{n+1}}\, f(\zeta_1, \zeta_2),$$

$$m!n!c_{mn} = \left(\frac{\partial^{m+n} f(z_1, z_2)}{\partial z_1^m\, \partial z_2^n} \right)_{\substack{z_1 = z_1^0 \\ z_2 = z_2^0}}.$$

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