Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 23, 2009

ON EXISTENCE AND UNIQUENESS OF SOLUTION OF GENERAL SECOND ORDER ELLIPTIC TYPE DIFFERENTIAL EQUATION IN HILBERT SPACE

Dochviri B., Purtukhia O., Sokhadze G.

Abstract. General type of elliptic differential equation in Hilbert space is considered. Theorem of existence and uniqueness is given. Probabilistic representation of the solution is given. The forward and backward stochastic differential equations technics is used.

Keywords and phrases: Elliptic equations, forward backward stochastic equations, probabilistic representation of solution.

AMS subject classification (2000): 60H10; 60H3X.

The basic problem of this paper is the possibility of probabilistic representation of solution of general second order elliptic type differential equation in Hilbert space. The equations of this type we have considered in more general case than usual. As elliptical parts we consider the operators obtained as superposition of logarithmical gradients of given measure and differential operators. Such kind of possibilities were considered by Belopolskaya J.I. and Daletskyi J.L. in [1]. The well developed theory of logarithmical gradient (see Bogachev V.V. [2] and the references there in) and theories of propabilistic representation of the solutions of parabolic and elliptic equations allow us to solve this problem.

Let $H_+ \subset H \subset H_-$ equipped Hilbert space with Hilbert-Schmidt embeding. The duality between H_- and H_+ denoted by $\langle u, k\rangle$, $u \in H_-$, $k \in H_+$. The scalar and norms products in this spaces we denote as $(h,k)_+, ||h||_+, h,k \in H_+$; $(h,k), ||h||, h,k \in H$ and $(h,k)_-, ||h||_-, h,k \in H_-$ respectively. In case when $u \in H_-, k \in H_+$, the duality $\langle u, k\rangle$ we'll write as the scalar product in $H : \langle u, k\rangle = (u,k)$. The Borel σ-algebra in topological space K is denoted by $\Re(K)$. Let μ be the measure on $\Re(H_-)$. We say, that μ has a logarithmical derivative $\beta_\mu(x,h) = \langle \lambda(x), h\rangle$, along of constant directions $h \in H_+$, if for any test functional $\varphi \in C_b^1(H_-)$ the formula of integration by part is holds:

$$\int_{H_-} (\nabla\varphi(x), h)\mu(dx) = -\int_{H_-} \varphi(x)\beta_\mu(x,h)\mu(dx). \tag{1}$$

In this case we write $\mu \in \Xi^0(H_-)$ and function $\lambda(x)$ we call as vector logarithmical derivative of measure μ. If relation (1) is fulfilled when instead of h in (1) is substituted vector field $z(x) : H_- \to H_+$ then we say that measure has logarithmical derivative along z. Let $C_1(H_-, H_-, H_+)$ is the class of vector fields defined on H_- with values in H_- and continuously differentiable along H_+. Let $\sigma_1(H_+)$ denotes the space of vector fields $z(x) : H_- \to H_+$ from the class $C_1(H_-, H_-, H_+)$ with the norm

$$\|z\|_1 = \sup_{x\in H_-} \{\|z'(x)\|_{L(H_-,H_+)}, \|z(x)\|_-\} < \infty.$$

It is known (see [1]), that if $\mu \in \Xi^1(H_-)$ and $z \in \sigma_1(H_+)$, then μ has logarithmical gradient along z and the following formula is true

$$\beta_\mu(z(x), x) = (\lambda(x), z(x)) + tr_H z'(x). \tag{2}$$

Let $\Xi^1(H_-, H) \subset \Xi^0(H_-)$ denotes the class of measures with smooth along H logarithmical derivative $\lambda(x)$ such, that $\lambda'(x) \in L(H)$ and
$Vrai \sup\limits_{x\in H_-} \|\lambda'(x)\|_{L(H)} < \infty$. Let introduce the norm

$$\sigma_{1,2}(z) = \left\{ \int_{H_-} [\|z(x)\|^2 + tr_H z'(x) z'(x)^*]\mu(dx) \right\}^{\frac{1}{2}}$$

in the space $C_1(H_-, H_-, H)$.

The completing of $C_1(H_-, H_-, H)$ by this norm, denoted by $H_1^2(H_-, H, \mu)$, is Hilbert space. If $L_1(H_-, \mu)$ is the space of μ-integrable functions, then we obtain, that linear differential operator

$$\beta_\mu : z \to (z(x), \lambda(x)) + tr_H(z'(x)))$$

realizes continuous mapping

$$\beta_\mu : \sigma_1(H_+) \to L_1(H_-, \mu).$$

Moreover we have

$$\begin{aligned} &\textstyle\int_{H_-} |\beta_\mu(z,x)|^2 \mu(dx) = -\int_{H_-} \{tr_H(z'(x))^2 \\ &-(\lambda'(x) z(x), z(x))\}\mu(dx) \le c \cdot \sigma_{1,2}^2(z). \end{aligned} \tag{3}$$

Hence β_μ continuously can be extended on whole $H_1^2(H_-, H, \mu)$. This extension we denote by $\beta_\mu(z, x)$ too. So, if $\mu \subset \Xi^1(H_-, H)$ and $z \subset H_1^2(H_-, H, \mu)$, then measure μ is differentiable along z, its logarithmical derivative $\beta_\mu \in L_2(H_-, \mu)$ and estimation (3) is hold.

Note, that for kernel operator $z'(x)$ the formula (2) is fulfilled, but the logarithmical gradient exists even if $z'(x)$ is Hilbert-Schmidt operator.

Consider the elliptic differential equation

$$(Lu)(x) + F(x, u(x), (\nabla u)\sigma) = 0, \tag{4}$$

where $(L\varphi)(x) = \frac{1}{2}\beta_\mu(x, \sigma\sigma^*\nabla\varphi(x)) + \langle Ax, \nabla\varphi(x)\rangle + \langle b(x), \nabla\varphi(x)\rangle$, A is the linear operator, σ is a linear operator such, that $\sigma\sigma^*$ represents Hilbert-Schmidt operator, $F(x, y, z)$-function on $H_- \times R \times L(H_-, H)$. Our aim is to find the conditions for existence and uniqueness of weak solutions for (4) and connect this solution to solution of corresponding stochastic differential equation (or system).

For this reason we make regularization of equation (4) and enstead of σ in (4) substitute $\sigma\theta_n$, where $\theta_n, n = 1, 2, ...$ the sequence of Hilbert-Schmidt operators in H which converges to unit operator I (under the operator norm). Then (4) has the form

$$\begin{aligned} &\tfrac{1}{2} tr_H \sigma\theta_n\theta_n^*\sigma\nabla^2 u_n + (Ax + \tfrac{1}{2}\lambda(x) + b(x), \sigma\theta_n\theta_n^*\sigma\nabla u_n) + \\ &F(x, u(x), (\nabla u_n)\sigma\theta_n) = 0. \end{aligned} \tag{5}$$

Let us consider forward stochastic differential equation

$$dX_t = AX_t dt + (\frac{1}{2}\sigma\theta_n\theta_n^*\sigma^*\lambda(X_t) + b(X_t))dt + \sigma\theta_n dW_t, t \geq 0, X_0 = x \tag{6}$$

and backward stochastic differential equation

$$dY_t^x = F(X_t^x, Y_t^x, Z_t^x)dt - Z_t^x dW_t, 0 \leq t \leq T < \infty, Y_t^x = x \tag{7}$$

And suppose, that the following conditions are fulfilled:

1) A is unbounded linear operator with domain $D(A) \subset H$, which is generator of strongly continuous semigroup $e^{tA}, t \geq 0$ in H. Moreover $\exists m > 0, a > 0$ such that $||e^{tA}|| \leq me^{at}, t \geq 0$;

2) $b : H_- \to H$ and exists the constant $L > 0$ such, that is valid the following inequality

$$||b(x) - b(y)|| \leq L||x - y||.$$

Furthermore $b(\cdot) \in C_1(H_-, H)$;

3) σ is a linear continuous operator $e^{tA}\sigma \in L_2(H_-, H)$ and

$$||e^{tA}\sigma||_{L_2(H_-,H)} \leq Lt^{-\gamma}e^{at}$$

for some $L > 0$ and $\gamma \in [0, 1/2)$;

4) Operators $A + \frac{1}{2}\sigma\sigma^*\lambda'_x(x) + b'_x(x)$ are dissipative in the following sense: for $\forall x \in H, y \in D(A)$ we have

$$(Ay, y) + \frac{1}{2}(\sigma\sigma^*\lambda'_x(x)y, y)) + (b'_x(x)y, y) \leq 0;$$

5) The function $F(\cdot, \cdot, \cdot)$ is determined on $H \times R \times H_-$, takes real values and has derivatives over all arguments;

6) There exists such constant $C > 0$, that

$$|F(x, y, z)| \leq C(1 + |y| + |z|^2);$$

7) $|| \bigtriangledown_x F(x, y, z)|| \leq C$;

8) $||\nabla_z F(x, y, z)||_- \leq C(1 + ||z||_-)$;

9) $|\nabla_y F(x, y, z)| \leq C(1 + ||z||_-^2)$;

10) F represents the monotone function, with constant monotony $\rho > 0$, in the following sense: for $\forall x \in H, y, y' \in R, z \in H_-$ we have

$$(y - y')(F(x, y, z) - F(x, y', z)) \leq -\rho|y - y'|^2.$$

It follows from [7,8], that in conditions 1)-10) the system of equations (6),(7) has unique solution $(X^x(n), Y^x(n), Z^x(n))$, such, that:
a) For any $p \in [2, \infty)$ and $T > 0$ we have $X^x_{(n)} \in L^p(\Omega; C(0, T); H)$ and

$$E[\sup_{t\in[0,T]} |X_t^x|^p] \leq C(1 + \|x\|)^p,$$

where C is the constant depended on p, γ, T, L, a and m;
b) Y^x is continuous process adapted with $\{\mathcal{F}_t\}_{t\geq 0} = \sigma\{W_t, 0 \leq t \leq T\}$ and bounded by constant $\frac{C}{\rho}$;
c) Z^x represents the progressively measurable process and for any $\varepsilon > 0$

$$E\int_0^\infty e^{-2\varepsilon t}\|Z_t\|^2 dt < \infty;$$

d) For each $T > 0, p \geq 1$ the mapping $x \to (Y^x|_{[0,T]}, Z^x|_{[0,T]})$ is continuous from H in $L^p(\Omega; C(0,T), R) \times L^p(\Omega; L^2(0,T; H_-))$.

Theorem. *In the equipped separable real Hilbert space $H_+ \subset H \subset H_-$ consider elliptic differential equation (4) and conditions 1)-10) are fulfilled. If $\mu \in \Xi^1(H_-, H)$ and $\sigma\sigma^*$ represents Hilbert-Schmidt operator, then there exists unique solution of the equation (4). Moreover there exists unique solution of forward-backward stochastic differential system (6)-(7) - $(X^x(n), Y^x(n), Z^x(n))$, which satisfies properties a), b), c) and d) for each n. This solution converges for $n \to \infty$, $\theta_n \to I$, $\theta_n \in L_2(H,H)$ and the solution of (4) can be represented in form $u(x) = Y_0^x(\infty)$.*

Scheme of proof. In the beginning let us show, that, based on construction in Theorem 5.2 from [7], the equation (5) has unique solution and the following representation is valid $u_n(x) = Y_0^x(n)$. Using property b) let us show, that this value is uniformly bounded by n and therefore there exists sequence $\{n_k\}$ such, that for $k \to \infty$ there exists the limit $Y_0^x(n_k)$. Denote this limit by $Y_0^x(\infty)$. Let us show, that it satisfies the equation (4). It is easy to see, that for $\theta_n \to I$ we have

$$tr_H \sigma\theta_n\theta_n^*\sigma^*\nabla^2 u + (\lambda(x), \sigma\theta_n\theta_n^*\sigma^*\nabla u) \to \beta_\mu(\sigma\sigma^*\nabla u, x)$$

(for instance it follows from (3)). Therefore using properties 6)-10) we obtain, that $u(x) - u_n(x)$ converges to 0 in respective space.

Remark. Analogously we can show, that $Y_t^x = u(X_t^x)$ and $Z_t^x = \nabla u(X_t^x)\sigma$.

REFERENCES

1. Daletskyi J.L., Belopolskaya J.I. Stochastic Equations and the Differential Geometry. (Russian) *Kiev, "Visha shkola"*, 1989.

2. Bogachev V.V. Differentiable Measures and the Malliavin Calculus. (Russian) *Moscow RCHD*, 2008.

3. Daletskyi J.L. Infinite dimensional elliptic operators and connected parabolic equations. (Russian) *UMN*, **22**, 4 (1967), 3-54.

4. Belopolskaya J.I., Daletskyi J.L. The Markov processes and connected nonlinear parabolic systems. (Russian) *Dokl. AN SSSR*, **250**, 3 (1980), 521-524.

5. Pardoux E., Peng S. G. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, **14** (1990), 55-61.

6. Constantin A. Global existence of solutions for perturbed differential equations. *Anali di Mat. Pura ed Appl. Serie IV*, **CLXVIII**, (1995), 237-299.

7. Briand P., Confortola F. Quadratic BSDEs with random terminal time and elliptic PDEs in infinite dimension. *Electronic J. of Probability.* **13**, Paper no. 54 (2008), 1529-1561.

8. Fuhrman M., Tessitore G. Nonlinear Kolmogorov equation in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. *Ann. Probab.*, **30**(2002), 1397-1465.

Received 23.05.2009; revised 29.10.2009; accepted 30.12.2009.

Authors' address:

B. Dochviri, O. Purtukhia, G. Sokhadze
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: omar.purtukhia@tsu.ge
giasokhi1@i.ua