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ON THE COMPARISON OF PRICES IN THE DISCRETE KALMAN–BUCY MODEL

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Abstract. An optimal stopping problem is considered using incomplete data in the discrete Kalman–Bucy model of partially observable random sequences. It is shown that the price in the problem with incomplete data does not exceed the price in the problem with complete data and tends to the latter price when a small perturbation parameter in the observable process tend to zero.

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1. Let us consider on some probability space $(\Omega, \mathcal{F}, P)$ the Kalman–Bucy discrete model for a partially observable stochastic sequence (θ_n, ξ_n), $n = 0, 1, \dots, N$, $N < \infty$, where

$$\theta_{n+1} = a_0(n) + a_1(n)\theta_n + b(n)\eta_{n+1}, \tag{1}$$
$$\xi_n = A_0(n) + A_1(n)\theta_n + \varepsilon\widetilde{\eta}_n. \tag{2}$$

The coefficients $a_i(n)$, $A_i(n)$, $i = 0, 1$, $b(n)$, are finite nonrandom functions, $\varepsilon \geq 0$ is a constant, η_n and $\widetilde{\eta}_n$ are independent standard normal random variables. It is assumed that θ_n is an unobserved and ξ_n is an observed sequence, and $A_1(n) \neq 0$ [1].

We consider the following gain function

$$g(n, x) = \sum_{i=0}^{k} f_i(n)x^k, \tag{3}$$

where $f_i(n) \geq 0$, $i = 0, 1, \dots, k$, $n = 0, 1, \dots, N$, and define the payoffs by the equalities

$$s^0 = \sup_{\tau \in \mathfrak{M}^\theta} Eg(\tau, \theta_\tau),$$

$$s^\varepsilon = \sup_{\tau \in \mathfrak{M}^\varepsilon} Eg(\tau, \theta_\tau),$$

where $\mathfrak{M}^\theta$ and $\mathfrak{M}^\varepsilon$ are the classes of stopping times with respect to the families of σ-algebras $\mathcal{F}_n^\theta = \sigma\{\theta_i, i \leq n\}$ and $\mathcal{F}_n^\varepsilon = \sigma\{\xi_i, i \leq n\}$, $\tau = 0, 1, \dots, N$. The price s^0 corresponds to the case of complete observation, and s^ε to the case of incomplete observation of a process θ_n.

In the case of complete observation, the basic problems of the stopping time theory of a sequence θ_n are finding a price s^0 and determining its structure, and also finding an optimal stopping time moment τ^* for which we have $s^0 = Eg(\tau^*, \theta_{\tau^*})$.

In the case of incomplete observation, in addition to these problems we must also write the price s^ε, for which we use some completely observable process (reduction problem) and prove the convergence of the price s^ε to the price s^0 as $\varepsilon \to 0$ (price convergence problem) [4]. To solve these problems, we use filtration equations [1], by means of which the difference $s^0 - s^\varepsilon$ can be estimated in terms of the parameter ε.

In this paper, we solve the price reduction and convergence problems for scheme (1), (2) and the gain function (3).

2. Let us introduce the values

$$m_n = E(\theta_n|\mathcal{F}_n^\xi), \quad \gamma_n = E\big[(\theta_n - m_n)^2|\mathcal{F}_n^\xi\big].$$

Lemma 1. *The following expression of a payoff function*

$$s^\varepsilon = \sup_{\tau\in\mathfrak{M}^\xi} Eg(\tau, m_\tau) = \sup_{\tau\in\mathfrak{M}^{\widetilde{\theta}}} Eg(\tau, \widetilde{\theta}\tau)$$

holds, where

$$\widetilde{\theta}_{n+1} = a_0(n) + a_1(n)\widetilde{\theta}_n + \beta(n)\eta_{n+1},$$

$$\beta(n) = \frac{b^2(n)A_1(n+1) + a_1^2(n)A_1(n+1)\gamma_n}{\sqrt{b^2(n)A_1^2(n+1) + a_1^2(n)A_1^2(n+1)\gamma_n + \varepsilon}}\,.$$

The proof of this lemma is analogous to that of Theorem 3 in [4].

Lemma 2. *The following estimate*

$$\gamma_n \le c_1\varepsilon$$

holds, where the constant $c_1 > 0$ is defined in explicit form.

The proof of this lemma is obtained using the following equation for γ_n [1]

$$\gamma_{n+1} = a_1^2(n)\gamma(n) + b^2(n) - \beta^2(n).$$

Assume $\mathcal{F}_n^* = \sigma\{\theta_i, \widetilde{\eta}_i;\ i = 1, \dots, n\}$ and denote by $\mathfrak{M}^*$ the class of stopping times with respect to a family of the σ-algebra $\mathcal{F}_n^*$.

Lemma 3. *The sequence $(x_n, \mathcal{F}_n^*)$, where $x_n = g(n, \theta_n)$, is a Markovian chain.*

The proof of this lemma is as follows. It can be easily seen that a random sequence $(x_n, \mathcal{F}_n^\theta)$ is a Markovian chain ([3], [Theorem 3.1]). Let a random value ξ be measurable with respect to the σ-algebra $\sigma\{x_i; i = 1, \dots, n\}$. Then we have

$$E\big(\xi|\theta_1, \dots, \theta_n, \widetilde{\eta}_1, \dots, \widetilde{\eta}_n\big) = E(\eta|\theta_n),$$

from which we obtain the assertion of Lemma 3 by virtue of Theorem 1.11 [1].

Now let us introduce the price

$$s^* = \sup_{\tau\in\mathfrak{M}^*} Eg(\tau, \theta\tau).$$

Lemma 4. *The class $\mathfrak{M}^*$ is the class of randomized stopping times.*

The proof of this lemma is obtained by means of Theorem 21 from [2].

The main result of this paper is the following

Theorem. *Let a partially observed random sequence* (θ_n, ξ_n), $n = 0, 1, \dots, N < \infty$, *be defined by the recurrent relations* (1), (2) *and a gain function have form* (3). *Then the following estimate*

$$0 \leq s^0 - s^\varepsilon \leq c_2 \varepsilon$$

holds, where the constant $c_2 > 0$ *is defined in explicit form.*

The proof of this theorem is based on Lemmas 1–4.

R E F E R E N C E S

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