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THE ELASTIC EQUILIBRIUM OF INFINITE PLATE CONTAINING RADIAL CRACKS ORIGINATING AT THE BOUNDARY OF INTERNAL CIRCULAR HOLE

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Abstract. Using the complex variable method of Muskhelishvili solution is obtained for two-dimensional boundary value problem of elastic equilibrium of infinite homogeneous isotropic body having circular hole with radial cracks of finite length.

Keywords and phrases: Boundary value problem, complex variable method, radial crack, mapping function.

AMS subject classification: 74B05.

We shall consider the solution of the class of plane problems in elasticity corresponding to a distribution of radial cracks, equal and finite in length, originating at the boundary surface of a circular hole in an infinite plate under the load system shown in Figure 1. The geometry of the internal boundary γ, can be conveniently described by considering the plate as the complex Z plane, where $Z = x + iy = re^{i\theta}$ Then, if the center of the hole is chosen as Z=0, we specify that radial cracks of equal length L, lie along θ=0, $2\pi/\kappa$,...,$(\kappa-1)2\pi/\kappa$, where $\kappa\geq 1$ is an integer which specifies the number of cracks.

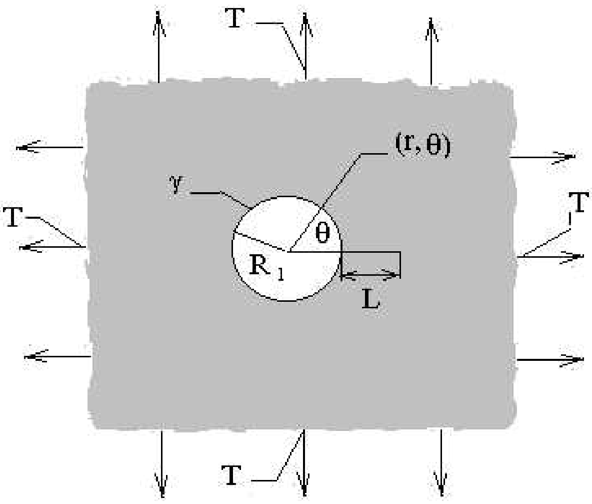


Fig.1. Geometry and loading for the single crack (k=1).

Due to the irregular geometry of the internal boundary, it would appear that the problem described above can be most conveniently handled by the complex variable method of Muskhelishvili [1]. This method depends upon the representation of the well-known Airy's stress function $U(x,y)$, in terms of two analytic functions of the complex variable Z, namely φ(Z) and ψ(Z) [1], where

$$U\left(x,y\right)=Re\left[\bar{Z}\varphi\left(Z\right)+\int_{Z}\psi\left(Z\right)dZ\right]. \tag{1}$$

With this representation, the stress components in rectangular coordinates can be written as:

$$\sigma_{y}+\sigma_{x}=2\left[\varphi'\left(Z\right)+\overline{\varphi'\left(Z\right)}\right]=4Re\left[\varphi'\left(Z\right)\right], \tag{2}$$

$$\sigma_{y}-\sigma_{x}+2i\tau_{xy}=2\left[\overline{Z}\varphi''\left(Z\right)+\psi'\left(Z\right)\right]. \tag{3}$$

It is convenient for the purpose of enforcing boundary conditions to introduce an auxiliary complex plane, the ζ plane, such that the unit circle $\zeta=\delta=e^{i\beta}$ (where β is used here to denote angular measure in the ζ plane), and its exterior are mapped into γ and its exterior, respectively, by the analytic function

$$Z=\omega\left(\zeta\right). \tag{4}$$

The stress functions $\varphi(Z)$ and $\psi(Z)$ will be considered as function of the parameter ζ. The necessity for introducing considerable new notation can be avoided by designating $\varphi(Z)$=$\varphi[\omega(\zeta)]$ as φ (ζ) etc., which leads to such relationships as $\varphi\prime(Z)=\varphi\prime(\zeta)/\varpi\prime(\zeta)$ etc.

Above-described problem requires the determination of the functions $\varphi(\zeta)$ and $\psi(\zeta)$ which are analytic for $|\zeta|>1$ (with the exception of the point at infinity) and lead to the proper loading conditions at infinity and on the internal boundary γ. The forms of $\varphi(\zeta)$ and $\psi(\zeta)$ can be chosen a priori to yield the proper loading condition at infinity. The condition that γ be load-free can be written as [2]

$$\varphi\left(\delta\right)+\omega\left(\delta\right)\varphi'\left(\delta\right)\Big/\overline{\omega'\left(\delta\right)}+\overline{\psi\left(\delta\right)}=0. \tag{5}$$

The mapping function (4) is the product transformation and can be expressed in differential form as [2]

$$dZ/Z=\left(1-\zeta^{-k}\right)d\zeta\Big/\left(\zeta\left(1+2\in\zeta^{-k}+\zeta^{-2k}\right)^{\frac{1}{2}}\right). \tag{6}$$

In (6) $\in$ is a real parameter such that $0\leq|\in|\leq1$ and the denominator is considered positive at ζ=1 in order to define the desired branch of the multivalued mapping function. By varying $\in$, the crack depth can be adjusted to assigned values [2].

For purposes of the subsequent stress analysis it is desirable to find a series representation of (6) converging on and exterior to the unit circle. The form of such a series is evidently

$$Z=\omega\left(\zeta\right)=C\left[\zeta+\sum_{n=1}^{\infty}A_{n}\zeta^{1-kn}\right], \tag{7}$$

where the A_n's are real and they may be obtained numerically from simple recursive formulae determined by expanding both sides of (6) in series form and equating coefficients of equal powers of ζ

$$A_1 = \frac{\in +1}{k}; \quad A_2 = \frac{(1-k)\in +1}{k} A_1 + \frac{(1-k)^2 - 1}{4k} A_1^2;$$
$$A_3 = -\frac{1}{3}\left\{ A_1 - \frac{2\left[(1-2k)\in +1\right]}{k} A_2 - \frac{(1-k)(1-2k)-1}{k} A_1 A_2 - \frac{(1-k)^2 \in +1}{k} A_1^2 \right\};$$
$$A_m = -\frac{1}{m}\left\{ (m-2) A_{m-2} - \frac{2\left[(1-(m-1)k)\in +1\right]}{k} A_{m-1} - \right.$$
$$-\frac{1}{2k}\sum_{l=1}^{m-1}\left\{(1-lk)\left[1-(m-l)k\right]-1\right\} A_l A_{m-l} -$$
$$-\frac{1}{k}\sum_{l=1}^{m-2}\left\{(1-lk)\left[1-(m-l-1)k\right]\in +1\right\} A_l A_{m-l-1} -$$
$$\left. -\frac{1}{k}\sum_{l=1}^{m-3}\left\{(1-lk)\left[1-(m-l-2)k\right]-1\right\} A_l A_{m-l-2}; \right\}, m = 4, 5, 6, \cdots.$$

The convergence of (7) on and exterior to the unit circle can be studied by examining the coefficients A_n. It can be shown that $\lim_{n\to\infty} A_n = 0$, thus, using a well-known theorem found in [3], the series (7) converges exterior to the unit circle and at all points on the unit circle except at the roots of $\zeta^{2k} + 2\in\zeta^k + 1 = 0$.

The existence of cusps at locations corresponding to the crack roots is ensured in polynomial mapping approximations [2]. Due to the convergence of (7) at all but a finite number of points on the unit circle, suitable polynomial approximations $Z = \omega(\zeta) = C\left[\zeta + \sum_{n=1}^{N} d_n \zeta^{1-kn}\right]$ can be obtained by setting $d_n \cong A_n$.

Consider now the case of uniform tension at infinity illustrated in Figure 1. It can easily be shown that the loading condition $\sigma_x = \sigma_y = T$ on $|Z|$=R, where R is very large is satisfied by choosing $\varphi(\zeta)$ and $\psi(\zeta)$ such that they approach $CT\zeta/2$ and $CT\gamma_0\zeta^{-1}$ (where $\gamma_0 = -\left[1 + 2\sum_{n=1}^{N} d_n \alpha_n (1-2n)\right]$), respectively, for large $|\zeta|$. Therefore, let us assume that $\varphi(\zeta)$ is a polynomial of the form [2]

$$\varphi(\zeta) = CT\left[\zeta/2 + \sum_{n=1}^{N} \alpha_n \zeta^{1-kn}\right]. \tag{8}$$

Next, we write the boundary condition (5) as

$$\omega'(\delta)\,\psi(\delta) = -\omega'(\delta)\,\overline{\varphi(\delta)} - \overline{\omega(\delta)}\varphi'(\delta). \tag{9}$$

The function $\varpi\prime(\zeta)\psi(\zeta)$ is analytic exterior to the unit circle and with $\varphi(\zeta)$ assumed as (8), is given as a continuous function on the unit circle by (9). Thus, if the coefficients

α_n can be chosen so that the coefficients of all positive powers of ζ in the Laurent expansion of

$$\omega'(\zeta)\,\varphi(1/\zeta)+\omega(1/\zeta)\,\varphi'(\zeta) \tag{10}$$

vanish, we can determine $\psi(\zeta)$ explicitly. By multiplying both sides of (9) by $1/2\pi i(\delta-\zeta)$ and integrating around the unit circle, we obtain by a well-known theorem [4]

$$\omega'(\zeta)\,\psi(\zeta)=-\omega'(\zeta)\,\varphi(1/\zeta)-\omega(1/\zeta)\,\varphi'(\zeta)\,.$$

It is interesting to note that the cups roots are reflected by singularities in $\psi(\zeta)$ in the form of simple poles.

In conclusion it is necessary to verify that the coefficients α_n can be determined to meet the condition set forth above. If the coefficients of all positive powers of ζ in the Laurent expansion of (10) are equated to zero, the following system of linear simultaneous equations results:

$$\alpha_p+\sum_{n=1}^{N-p}\alpha_{p+n}d_n\,(1-nk)+\sum_{n=1}^{N-p}d_{p+n}\alpha_n\,(1-nk)+d_p/2=0,\quad p=1,2,\cdots,N.$$

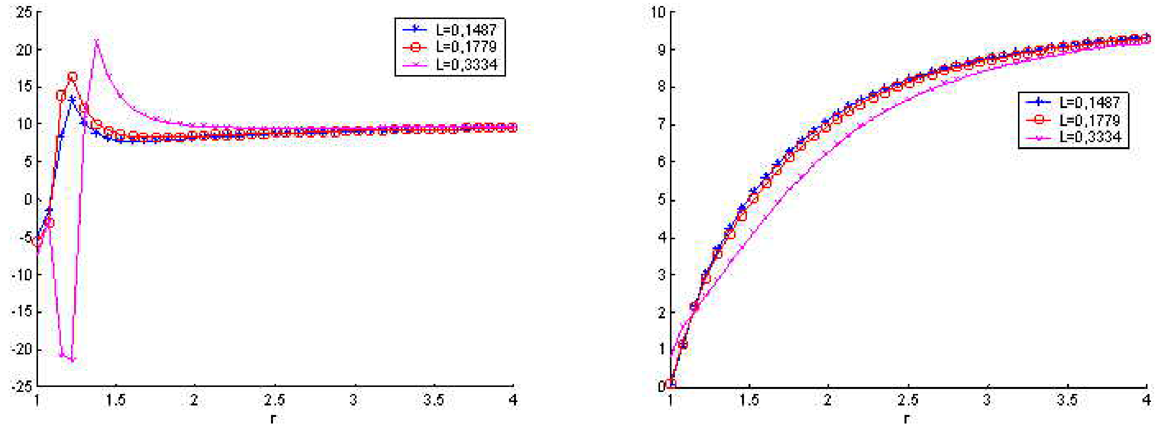
The hitherto unspecified constant C occurring in the mapping function will mow be chosen so that the radius of the circular hole in the physical plane is the unit of length. Thus, if $\delta=\delta_1$ is that point on the unit circle in the ζ plane which corresponds to the function of the crack and the circle in the Z plane, then C is chosen so that $|\omega(\delta_1)|=1$.

In polar coordinates the components of stress and displacement σ_r, σ_θ, $\tau_{r\theta}$, U_r, and U_θ can be expressed in terms of the original stress function $\varphi(Z)$ and $\psi(Z)$ defined by (2) and (3) form the relations

$$\sigma_r-i\tau_{r\theta}=\varphi'(Z)+\overline{\varphi'(Z)}-e^{2i\theta}\left[\overline{Z}\varphi''(Z)+\psi'(Z)\right],$$
$$\sigma_\vartheta+i\tau_{r\theta}=\varphi'(Z)+\overline{\varphi'(Z)}+e^{2i\theta}\left[\overline{Z}\varphi''(Z)+\psi'(Z)\right],$$
$$2\mu\,(U_r+iU_\theta)=e^{-i\theta}\left[\eta\varphi(Z)-Z\overline{\varphi'(Z)}-\overline{\psi(Z)}\right],$$

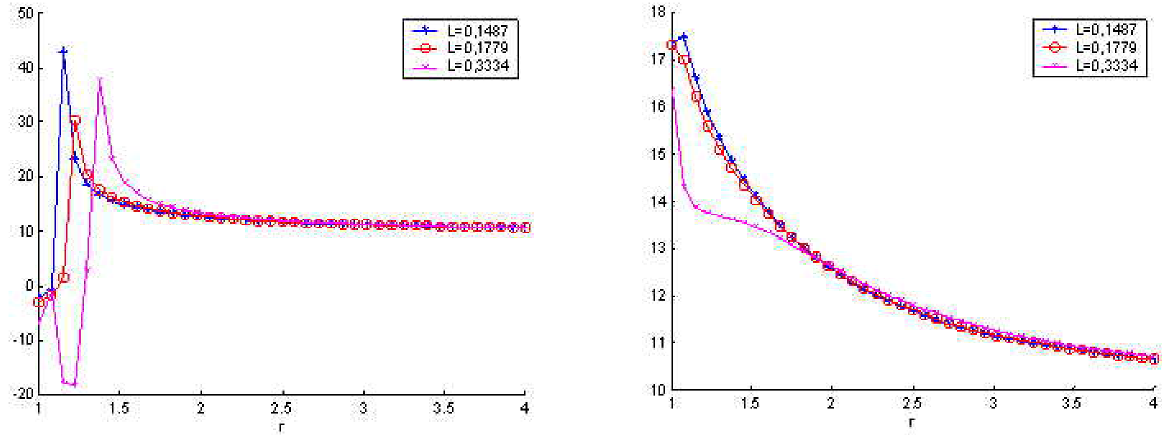
where $\mu=E/2\,(1+\nu)$, $\eta=(3-\nu)\,/\,(1+\nu)$, E – Young's Modulus, ν - Poisson's ratio.

At the characteristic points of the considered domain we have obtained numerical results of normal and shearing stresses for three values of crack length L=0,1779; 0,1487; 0,3334; R_1=1m, $T=10kg/cm^2$, $\theta=\pi/180;\pi/4$, $R_1<r<4$, and constructed graphs (Fig.2, Fig.3).



a) b)

Fig.2. Normal stresses σ_r for $R_1 < r < 4$ and a) $\theta = \pi/180$, b) $\theta = \pi/4$



a) b)

Fig.3. Shearing stresses σ_θ for $R_1 < r < 4$ and a) $\theta = \pi/180$, b) $\theta = \pi/4$

At the end it is important to explain that results received for one crack by the boundary element method [5] and the complex variable method are equivalent with possible exactness of technical works. Investigation for one and multiple cracks made by the boundary element method gets proper recommendations to underground constructions.

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