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ON SOME PROPERTIES OF STATISTICAL STRUCTURE

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Abstract. In the paper there is discussed statistical structures $(H, \mathcal{B}(H), \mu_a, a \in H)$ in Hilbert separable space. The necessary and sufficient conditions, for proving when the above mentioned statistical structure assumes optimal consistent estimate of parameters, are proved. There are formed weakly separable statistical structures for which there do not exist optimal consistent estimates.

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Let there is given (E, S) measurable space and on this space there is given $\{\mu_a, a \in A\}$ family of probability measures depended on $a \in A$ parameter. Let bring some definitions (see [1]–[6]).

Definition 1. The following object $\{E, S, \mu_a, a \in A\}$ called statistical structure connected with a stochastic system.

Definition 2. A statistical structure $\{E, S, \mu_a, a \in A\}$ connected with a stochastic system is called orthogonal (singular) if

$$(\forall\, i)\ (\forall\, j)\ \big(i \in A\ \&\ j \in A\ \&\ i \neq j \Longrightarrow \mu_i \perp \mu_j\big).$$

Definition 3. A statistical structure $\{E, S, \mu_a, a \in A\}$ connected with stochastic system is said to be weakly separable, if there exists a family of S-measurable sets $\{X_a, a \in A\}$ such that the relations

$$(\forall\, i)\ (\forall\, j)\ \big(i \in A\ \&\ j \in A\big) \Longrightarrow \mu_i(X_j) = \begin{cases} 0, & \text{if } \mathrm{i} \neq \mathrm{j}, \\ 1, & \text{if } \mathrm{i} = \mathrm{j} \end{cases}$$

are fulfilled.

Definition 4. A statistical structure $\{E, S, \mu_a, a \in A\}$ connected with a stochastic system is said to be strongly separable, if there exist pairwise disjoint S-measurable sets $\{X_a, a \in A\}$ such that the relation $(\forall\, i)\ \big(i \in A \Longrightarrow \mu_i(X_i) = 1\big)$ is fulfilled.

Definition 5. A statistical structure $\{E, S, \mu_a, a \in A\}$ connected with a stochastic system will be said to admit a consistent estimate of parameters if there exists a measurable map g of the space (E, S) in $(A, \mathcal{B}(A))$ such that $\mu_a\big(x :\ g(x) = a\big) = 1$ for each $a \in A$.

Let H is separable Hilbert space and $\mathcal{B}(H) - \sigma$-algebra formed by H Borel measures. Let mark the scale product (x, y) on H. It is obvious that the probability μ measures on $(H, \mathcal{B}(H))$ is given with typical function $\chi(z) = \int\limits_H e^{i(z,x)}\mu(dx)$. The typical function of Gaussian μ measure is given by the formula $\chi(z) = \exp\left\{i(a,z) - \frac{1}{2}(Bz,z)\right\}$, where $a \in H$ is approximate meaning and B is the correlating operator. Let us discuss Cauchy countable measures on $(H, \mathcal{B}(H))$, by different approximations $a_k \in H$ and same correlating operators. Let mark such Gaussian measures $\{\mu_{a_k}, k \in N\}$, so we have Gaussian statistical structure $\{H, \mathcal{B}(H), \mu_{a_k}, k \in N\}$.

Let us prove the following theorem.

Theorem. *In order to $\{H, \mathcal{B}(H), \mu_{a_k}, k \in N\}$ statistic structure form optimal estimates of $a_k \in H$ parameter, it is necessary and enough to fulfill the following conditions each $a_k \neq a_j$ $\sum_{i=1}^{\infty} \frac{(a_k - a_j, e_i)^2}{\lambda_i} = +\infty$, where $\{e_i\}_{i=1}^{\infty}$ are bases of B correlating operators and λ_i are the own meanings of B operator.*

Inevitability. Let $\{H, \mathcal{B}(H), \mu_{a_k}, k \in N\}$ statistical structure assumes optimal consistent estimates, so there exists such g measurable set $(H, \mathcal{B}(H))$ space in $(H, \mathcal{B}(H))$ space, that the following estimation $\mu_{a_k}\left\{x : g(x) = a_k\right\} = 1 \ \forall k \in N$, any $a_k \neq a_j$ for $k \neq j$ $\left\{x : g(x) = a_k\right\} \cap \left\{x : g(x) = a_j\right\} = \varnothing \ \forall k \neq j$. So there exist such disjoint $X_k = \{x : g(x) = a_k\}$. $\mathcal{B}(H)$ measurable sets that

$$\mu_{a_k}(X_k) = 1 \ \forall k \in N.$$

So, the statistical structure $\{H, \mathcal{B}(H), \mu_{a_k}, k \in N\}$ is strongly separable.

So, it is clear that for any $a_k \neq a_j$, there exist such $\mathcal{B}(H)$ measurable X_k sets that $\mu_{a_k}(X_k) = 1$, $\mu_{a_k}(X_j) = 0$ and $\mu_{a_j}(X_j) = 1$, $\mu_{a_j}(X_k) = 0$, then

$$\mu_{a_k}(H - X_k) = \mu_{a_k}(H) - \mu_{a_k}(X_k) = 1 - 1 = 0 \ \text{ and } \ \mu_{\mathrm{a_j}}(\mathrm{X_k}) = 0.$$

This means that Gaussian measures μ_{a_k} and μ_{a_j} are partly orthogonal, so $\{H, \mathcal{B}(H), \mu_{a_k}, k \in N\}$ statistical structure is orthogonal (see [5]). Proved the theorem. If Gaussian μ_{a_k} and μ_{a_j}, $a_k \neq a_j$ measures are orthogonal, then

$$\sum_{i=1}^{\infty} \frac{(a_k - a_i, e_i)^2}{\lambda_i} = +\infty$$

so the inevitabilities is proved.

Sufficiently. Let the following composition is fulfilled

$$\sum_{i=1}^{\infty} \frac{(a_k - a_i, e_i)^2}{\lambda_i} = +\infty.$$

Any $a_k \neq a_j$ according to Skorochod theorem again μ_{a_k} is orthogonal μ_{a_j}. So the statistical structure $\{H, \mathcal{B}(H), \mu_{a_k}, k \in N\}$ is orthogonal. Then there exist such $\mathcal{B}(H)$ measurable X_k sets that

$$\mu_{a_k}(\widetilde{X}_{jk}) = 0 \ \text{ and } \ \mu_{\mathrm{a_j}}(\mathrm{H} - \widetilde{\mathrm{X}}_{\mathrm{jk}}) = 0.$$

Let us discuss the sets

$$\widetilde{X}_j = \bigcup_{k \neq j} (H - \widetilde{X}_{jk}),$$

then

$$\mu_{a_j}(\widetilde{X}_j) = \mu_{a_j}\Big(\bigcup_{k\neq j}(H - \widetilde{X}_{jk})\Big) \leq \sum_{k\neq j}\mu_{a_j}(H - \widetilde{X}_{jk}) = 0$$

and

$$\mu_{a_k}(H - \widetilde{X}_j) = \mu_{a_k}\Big(H - \bigcup_{k\neq j}(H - \widetilde{X}_{jk})\Big) \leq \mu_{a_k}\big(H - (H - \widetilde{X}_{jk})\big) = 0$$

for each $a_k \neq a_j$.

So, $\{H, \mathcal{B}(H), \mu_{a_k}, k \in N\}$ statistical structure is weakly separable.

Let us discuss the following sets

$$X_i = \widetilde{X}_i - \widetilde{X}_i \bigcap \Big(\bigcup_{k\neq i}(H - \widetilde{X}_k)\Big)$$

it is clear that $X_i \cap X_j = \varnothing$ for each $i \neq j$ and $\mu_{a_k}(X_j) = 0$ for $k \neq j$, as $X_i \subset \widetilde{X}_i$ and it is known that $\mu_{a_k}(\widetilde{X}_j) = 0$. Let us show that $\mu_{a_j}(H - X_j) = 0$. Indeed,

$$H - X_j = H - \Big[\widetilde{X}_j - \widetilde{X}_j \bigcap \Big(\bigcup_{k\neq j}(H - \widetilde{X}_k)\Big)\Big].$$

So

$$\mu_{a_j}(H - X_j) = \mu_{a_j}\Big[H - \Big[\widetilde{X}_j - \widetilde{X}_j \bigcap \Big(\bigcup_{k\neq j}(H - \widetilde{X}_k)\Big)\Big]\Big] = 0.$$

So there exist such $\mathcal{B}(H)$ measurable disjunct sets X_j that

$$\mu_{a_j}(X_j) = 1 \ \ \forall j \in N.$$

Let define $g(x)$ map on $(H, \mathcal{B}(H))$ again on $(H, \mathcal{B}(H))$ by the following formula

$$g(x) = a_j, \ \text{ if } \ \mathrm{x \in X_j}, \ \ \mathrm{j = 1, 2, \ldots, n, \ldots},$$

where $a_j \in H \ \forall j \in N$. So it is obvious, that g $(H, \mathcal{B}(H))$ is measurable map and

$$\mu_{a_j}\big(x: \ g(x) = a_j\big) = 1 \ \ \forall j \in N.$$

So, $\{H, \mathcal{B}(H), \mu_{a_k}, k \in N\}$ statistical structure assumes optimal consistent estimates of parameters. So, the theorem is proved.

Note. It is obvious, the if $(H, \mathcal{B}(H), \mu_a, a \in H)$ statistical structure assumes optimal estimates, then this statistical structure is strongly separable and not viceversa.

For example. Let R be a real measurable space, and $L(R)$ be Lebesgue σ-algebra from R. Then there exist such one to one mapping $f: R \to R$, that f will not be $L(R)$ measurable. Let divide the segment $[-\frac{1}{2}, \frac{1}{2}]$ into classes as follows x and y dots are included in the some class, if only the difference of $x - y$ is a rational number. It is obvious that different classes are disjunctive. Let take one dot from each class and mark the multiplity of these dots with A. It is obvious that A multiply is not $L(R)$ measurable and card $A = c$ is continuum (see [6]).

As card $A=c$, then there exists one to one mapping of $f_1,\ f_1: A\to[0,1]$ so

$$f_1(A)=[0,1]\ \text{ as }\ \mathrm{A}\subset\Big[-\frac{1}{2},\frac{1}{2}\Big]\subset[-1,1].$$

Then it is obvious that card $([-1,1]-A)=c$ and so there exists beactive reflection of $f_2,\ f_2:([-1,1]-A)\to[-1,0]$ so

$$f_2=\big([-1,1]-A\big)=[-1,0].$$

Let

$$f(x)=\begin{cases} x, & \text{if } \ \mathrm{x}\in\mathrm{R}-[-1,1],\\ f_1, & \text{if } \ \mathrm{x}\in\mathrm{A},\\ f_2, & \text{if } \ \mathrm{x}\in[-1,1]-\mathrm{A},\end{cases}$$

f is measurable for $L(R)$ σ-algebra, as

$$f^{-1}[0,1]=f_1^{-1}[0,1]=A.$$

Let $(R,L(R))$ be a measurable space and f a one to one mapping of R on R, which is nonmeasurable for $L(R)$. For every $x\in R$ dot

$$\mu_x(E)=\begin{cases} 1, & \text{if } \ \mathrm{f(x)}\in\mathrm{E},\\ 0, & \text{if } \ \mathrm{f(x)}\notin\mathrm{E}.\end{cases}$$

It is obvious, that μ_x probability measures are gathered in $f(x)$ dots and statistical structure $(R,L(R),\mu_x,x\in R)$ is strongly statistical structure. It's power is continuum. Let us show, that for such strongly separable structures there is not optimal estimate.

Let, there exists an optimal estimate for such statistical structure. It means that there exists such $g:(R,L(R))\to(R,L(R))$ measurable mapping, that

$$\mu_x\big(x:\ g(x)=y\big)=1\ \ \forall\, y\in R.$$

The last equality shows us that

$$f(y)\in(x:\ g(x)=y).$$

So we have

$$g(f(y))=y\ \ \forall\, y\in R$$

on the one hand $f^{-1}(f(y))=y\ \forall\, y\in R$ on the other hand $g(f(y))=y\ \forall\, y\in R$ that is why we have $f^{-1}\circ f=g\circ f$ from have $f^{-1}=f\circ f^{-1}=g\circ f\circ f^{-1}$ it follows $f^{-1}=g$.

It is obvious, that if f is one to one mapping and it is nonmeasurable, then it is turned function is nonmeasurable too. But we got that f^{-1} is measurable. So, we have proved that there does not exist optimal estimate for such strongly separable statistical structure.

R E F E R E N C E S

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