TO VON KÁRMÁN TYPE SYSTEMS OF EQUATIONS FOR POROUS, PIEZO AND VISCO-ELASTIC PLATES

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Abstract. There are creating and justifying new 2D models of von Kármán type systems of nonlinear differential equations for porous, piezo and visco-elastic plates.

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One of the most principal objects in development of mechanics and mathematics is a system of nonlinear differential equations for elastic isotropic plate constructed by von Kármán . This system with corresponding boundary conditions represents the most essential part of the main manuals in elasticity theory. In spite of this in 1978 Truesdell expressed an idea about neediness of "Physical Soundness" of von Kármán system. This circumstance generated the problem of justification of von Kármán system. Afterwards this problem is studied by many authors, but with most attention it was investigated by Ciarlet [1]. In particular, he wrote:"the von Kármán equations may be given a full justification by means of the leading term of a formal asymptotic expansion" [1, p. 368]. This result obviously is not suffice for a justification of "Physical Soundness" of von Kármán system as representations by asymptotic expansions is dissimilar, leading terms are only coefficients of power series without any "Physical Soundness".

Based on the [2], the method of constructing such anisotropic inhomogeneous 2D nonlinear models of von Kármán –Mindlin-Reissner (KMR) type for binary mixture of poro, piezo and viscous elastic thin-walled structures with variable thickness is given, by means of which terms take quite determined "Physical Soundness". The corresponding variables are quantities with certain physical meaning: averaged components of the displacement vector, bending and twisting moments, shearing forces, rotation of normals, surface efforts. In addition the corresponding equations are constructed taking into account the conditions of equality of the main vector and moment to zero. By choosing parameters in the isotropic case from KMR type system (having a continuum power) the system as one of the possible models is obtained. The given method differs from the classical one by the fact, that according to the classical method, one of the equations of von Kármán system represents one of Saint-Venant's compatibility conditions, i.e. it's obtained on the basis of geometry and not taking into account the equilibrium equations. This remark is essential for dynamical problems. Further for isotropic and generalized transversal elastic plates in linear case from KMR the unified representation for all 2D BVP (considered in terms of planar expansions and rotations)

is obtained. So this report is devoted to problems of constructing the KMR type 2D BVP with respect to spatial variables for binary mixture of viscous-porous-elastic and piezo-electric and electrically conductive elastic thin-walled structures. At first will be introduced the nonlinear dynamic 3D (with respect to spatial variables) mathematical model for porous, piezo and viscous elastic media. At last we shall report the new iterative methods and numerical schemes for solving the corresponding BVP for 2D nonlinear systems of differential equations of KMR type.

Below we consider some simple (for obviousness) cases arising in the nonlinear problems of continuum mechanics and typical for seismology and structural mechanics too. Using methodology of [2], from ch.1 (in the case when thin-walled structure is an elastic isotropic homogeneous plate with constant thickness) we have the following nonlinear systems of PDEs of KMR type:

$$D\Delta^2\bar{u}_3 = \left(1 - \frac{h^2(1+2\gamma)(2-\nu)}{3(1-\nu)}\Delta\right)(g_3^+ - g_3^-) + 2h\left(1 - \frac{2h^2(1+2\gamma)}{3(1-\nu)}\Delta\right)[\bar{u}_3, \Phi^*]$$

$$+h(g_{3,\alpha}^+ - g_{3,\alpha}^-) - \int_{-h}^{h}\left(zf_{\alpha,\alpha} - \left(1 - \frac{1}{1-\nu}\Delta(h^2 - z^2)f_3\right)\right)dz + R_1[\bar{u}_3;\gamma], \quad (1)$$

$$\Delta^2\Phi^* = -\frac{E}{2}[\bar{u}_3, \bar{u}_3] + \frac{\nu}{2}\Delta(g_3^+ + g_3^-) + \frac{1+\nu}{2h}\bar{f}_{\alpha,\alpha} + R_2[\Phi^*], \quad (2)$$

$$Q_{\alpha 3} - \frac{1+2\gamma}{3}h^2\Delta Q_{\alpha 3} = -D\Delta\bar{u}_{3,\alpha} + \frac{h^2(1+2\gamma)}{3(1-\nu)}\partial_\alpha(g_3^+ - g_3^- + 2h(1+\nu))[\bar{u}_3, \Phi^*]$$

$$+h(g_\alpha^+ - g_\alpha^-) - \int_{-h}^{h}zf_\alpha dz + \frac{1+\nu}{2(1-\nu)}\int_{-h}^{h}(h^2 - z^2)f_{3,\alpha}dz + R_{2+\alpha}[Q_{\alpha,3};\gamma]. \quad (3)$$

The system (1)-(3) without reminder terms R gives 2D system of refined theories with control parameters γ. By choosing γ we got all refined theories and from other γ some new ones. Let us consider (1) equation underling the main members:

$$D'\Delta[w,\varphi] = D'\Delta([\Delta w, \varphi] + [w, \Delta\varphi] + 2[\partial_\alpha w, \partial_\alpha\varphi]), \quad (D' = 4h^3(1+2\gamma)/3(1-\nu)), \quad D\Delta^2 w.$$

By using for simplicity the typical relations as $\partial_{11}\varphi = \bar{\sigma}_{12}, \partial_{12}\varphi = -\bar{\sigma}_{12}, \partial_{22}\varphi = \bar{\sigma}_{11}$, the last expression may be rewritten in the following form:

$$D'\Delta[w,\varphi] = D'[(\bar{\sigma}_{11}\partial_{11}\Delta w + 2\bar{\sigma}_{12}\partial_{12}\Delta w + \bar{\sigma}_{22}\partial_{22}\Delta w) + (\partial_{11}w\Delta\bar{\sigma}_{11} + 2\partial_{12}w\Delta\sigma_{12}$$

$$+ \partial_{22}w\Delta\sigma_{22}) + 2(\bar{\sigma}_{11,\alpha}\partial_{11}w_{,\alpha} + 2\bar{\sigma}_{12,\alpha}\partial_{12}w_{,\alpha} + \bar{\sigma}_{22,\alpha}\partial_{22}w_{,\alpha})]. \quad (4)$$

The calculate and analysis by these expressions of a symbolical determinant show that the characteristic form of systems type (1) and (2) may be positive, negative or zero numbers as well as an arbitrary continuous function of x, y. Here we must remark that $ED' = 4(1+2\gamma)(1+\nu)D$, as so if $\{f\}$ denotes physical dimension of value f, it's evident $\{\Delta^2 w\} = \{\Delta[w, \Phi/E]\}$.

Thus, the first and second summands of (4) are defining the nonlinear wave processes for static cases. The structure of the third summand obviously corresponds to 2D soliton type solutions of Corteveg- de Vries or Kadomtsev-Petviashvili kind.

Analogous three-dimensional nonlinear model for anisotropic binary mixtures are presented in the works [3,4], which generalizes previously known model for porous-viscous-elastic binary mixtures. The constructed models together with certain independent scientific interest represent such form of spatial models, which allow not only to construct, but also to justify von KMR type systems as in the stationary, as well in nonstationary cases. Under justification we mean assumption of "Physical Soundness" to these models in view of Truesdell-Ciarlet (see for example details in [1, ch.5], [5]). As is known, even in case of isotropic elastic plate with constant thickness the subject of justification constituted an unsolved problem. The point is that von Kármán, Love, Timoshenko, Landau & Lifshits and others considered one of the compatibility conditions of Saint-Venant-Beltrami as one of the equations of the corresponding system of differential equations. This fact was verified also by Podio-Guidugli recently.

In the presented model we demonstrated a correct equation that is especially important for dynamic problems. The corresponding system in this case contains wave processes not only in the vertical, but also in the horizontal direction. The equation has the following form:

$$\left(\Delta^2 - \frac{1-\nu^2}{E}\rho\Delta\partial_{tt}\right)\Phi = -\frac{E}{2}[w,w] + \frac{\nu}{2}\left(\Delta - \frac{2\rho}{E}\partial_{tt}\right)(g_3^+ + g_3^-) + \frac{1+\nu}{2h}f_{\alpha,\alpha}. \quad (5)$$

The precision of the presented mathematical model is also conditioned by a new quantity, introduced in [2, ch.1], which describes an effect of boundary layer. Existence of this member not only explains a set of paradoxes in the two-dimensional elasticity theory (Babushka, Lukasievicz, Mazia, Saponjan), but also is very important for example for process of generating cracks and holes (details see in [2], ch.1, par. 3.3). Further, let us note that in works [4] equations of (5) type are constructed with respect to certain components of stress tensor by differentiation and summation of two differential equations. Also other equations of KMR type, which differ from (5) type equation, are equivalent to the system, where the order of each equation is not higher than two. For example, in the isotropic case, obviously, for coefficients we have $c_{aa} = \lambda^* + 2\mu$, $c_{66} = 2\mu$, $c_{12} = \lambda^*$, $c_{a6} = 0$, $\lambda^* = 2\lambda\mu(\lambda + 2\mu)a^{-1}$, λ and μ- are the Lame coefficients. Then the system (1.7) of [4] is presented in a form:

$$(\lambda^* + 2\mu)\partial_1\tau + \mu\partial_2\omega = \frac{1}{2h}\bar{f}_1 + \mu(\partial_1(\bar{u}_{3,2})^2 - \partial_2(\bar{u}_{3,1}\bar{u}_{3,2}))$$

$$-\frac{\lambda}{2h(\lambda + 2\mu)}\int\limits_{-h}^{h}\sigma_{33,1}dz,$$

$$-\mu\partial_1w + (\lambda^* + 2\mu)\partial_2\tau = \frac{1}{2h}\bar{f}_2 + \mu(\partial_2(\bar{u}_{3,1})^2 - \partial_1(\bar{u}_{3,1}\bar{u}_{3,2}))$$

$$-\frac{\lambda}{2h(\lambda + 2\mu)}\int\limits_{-h}^{h}\sigma_{33,2}dz,$$

$$(6)$$

where the functions: $\tau = \bar{\varepsilon}_{\alpha\alpha}$, $\omega = \bar{u}_{1,2} - \bar{u}_{2,1}$, correspond to plane expansion and rotation.

Thus, we intend to obtain the following results:

1. Nonlinear mathematical models for porous-viscous-elastic and elastic (with piezo-electric and electrically conductive processes) binary mixtures will be created and justified;

2. Questions of solvability of stationary and thermo-dynamical models (spatial case) will be investigated both in the linear and nonlinear anisotropic cases;

3. New two-dimensional with respect to spatial coordinates mathematical models of KMR type will be created and justified for porous-elastic binary mixtures when it represents a thin-walled structure; These models even in isotropic elastic case contain and justify (in sense of physical soundness) the well-known von Kármán system of DE for elastic plates; In these models we generalized Mauris Biot's models for anisotropic case and proved that the differential operator corresponding to spatial part is strong elliptical.For Biot-Darcy classical models corresponding symbolical determinant has singularity of second order;

4. Optimal models especially for nonhomogeneous systems of KMR type will be created and chosen without contracting a class of admissible solutions even in classical case;

5. Effective numerical methods will be constructed and justified; questions of convergence and error estimate will be studied for problems for thermo-porous elastic structures;

6. Questions of influence of new terms in the equation of form (5) will be investigated. Presence of these terms are very important, especially for seismic problems: in nonstationary problems these terms are of type $\partial_{tt}\Delta\Phi$, in stationary problems there are of type $\frac{\nu}{2}\Delta(q_3^+ + q_3^-)$.

Let us consider the problem of creating and justifying 2D mathematical models of KLM type for some dynamic nonlinear models of visco-elasticity when a solid structure is thin-walled one. We remind that this problem is important and open more than half century (see, for example, [5,6]). If using 3D nonlinear spatial system by [5] (see, p.119,

expression (12)) in the following form:

$$\partial I = \int_v \{(\epsilon_{ij} - \dot{\epsilon}_{ij} - \dot{\epsilon}''_{ij})\delta\dot{\sigma}_{ij} - \delta\dot{u}_k \frac{d}{dt}[\delta_{ik} + u_{k,i})\sigma_{ij}]_{,j} dV\}$$

$$\int_{S_s} (\dot{T}_i - \dot{\bar{T}}i)\delta\dot{u}_i dS - \int_{S_d} (\dot{u}_i - \dot{\bar{u}}_i)\delta\dot{T}_i dS,$$

where $T_i, \bar{T}_i$ are surface and prescribed surface tractions correspondingly (the other values as above have the same meanings). Evidently, from this functional follows, that the corresponding system of PDEs having the form same to system of spatial theory of elasticity:

$$\partial_i(\sigma_{ij} + \sigma_{kj}u_{i,k}) = f_i + \rho\partial_{tt}u_i$$

and necessary basic relations-initial and boundary conditions. Then the construct and justification of refined theories of KRM type didn't contain any principal difficulties and are realized similarly as in elastic case.

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