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OBSERVATION ON NUMERICAL SOLUTION OF NON LINEAR EQUATIONS SYSTEM

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Abstract. In this work there are giving observation and modification of numerical solution of some nonlinear system of equations. New methodology are considering and realizing for concrete example having practical meaning.

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Let us consider the system of non linear relationships

$$F_k\left(A_1,\ A_2,\ A_3,\dots A_n,\ A_{n+1}\right)=0,\ \ (k=1,2,3,\dots,n). \tag{1}$$

This system is unclosed because it contains n non linear relationships which depend of $(n+1)$ parameters. That's why it is impossible to unambiguously find the solution of the system. To find way out, let us consider one of parameters, for example, $A_{n+1}=P_0$ to be the conducting parameter change its value forcibly. Other n parameters will be leaded ones and their values will be defined like solution of the system of n non linear equations with the same number of unknowns

$$F_k\left(A_1,\ A_2,\ A3,\dots A_n,\ P_0\right)=0,\ \ (k=1,2,3,\dots,n). \tag{2}$$

The theories of solution of non linear equations system (2) are elaborated and there are many different iteration methods that exist for a long time. They make it possible to solve such systems if it's known the approximate value of solution, which ensures the convergence of iteration process. So, the essence of our interest is to discuss the method of definition of such approximate answer that make possible to increase probability of existing conditions for convergence of iteration process. It is of no importance whether it will be the method of Newton-Cantorovich or other one, though in further we will consider this method exactly as an example.

Before discussing the primal question, let us advance a remark. Not necessarily the last parameter to be the conducting parameter $A_{n+1}=P_0$. Any parameter may be conducting if parameters are renumbered. Moreover the number of parameters A_k may be equal to number of correlations in the dependences (1), i. e. n. At that additional parameter can be put in or even artificially created having guided by the heart of the problem or any other considerations which link the non linear and linear indeterminateness. We started thinking over the problem and publish our considerations about half century ago [1,2], and return to them last year [9], although we had never stopped to think on this matter [3,5 - 8]. Let us introduce the following matrix and vector notations:

$$F(A,P_0)=\begin{pmatrix} F_1(A,P_0)\\ F_2(A,P_0)\\ \cdots\cdots\\ F_{n-1}(A,P_0)\\ F_n(A,P_0)\end{pmatrix},\quad A^j=\begin{pmatrix} \left| \begin{array}{l} A_1,\\ A_2\\ \cdots\\ A_{n-1}\\ A_n \end{array}\right. \end{pmatrix},\quad \Delta A=\begin{pmatrix}\Delta A_1\\ \Delta A_2\\ \cdots\\ \Delta A_{n-1}\\ \Delta A_n\end{pmatrix},$$

$$W=\begin{pmatrix} B_1,1 & B_{1,2} & B_{1,3} & & B_{1,n-1} & B_{1,n}\\ B_2,1 & B_{2,2} & B_{2,3} & & B_{2,n-1} & B_{2,n}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ B_{n-1,1} & B_{n-1,2} & B_{n-1,3} & & B_{n-1,n-1} & B_{n-1,n}\\ B_{n,1} & B_{n,2} & B_{n,3} & & B_{n,n-1} & B_{n,n}\end{pmatrix}, \tag{3}$$

$$B_{k,m}=\frac{\partial F_k}{\partial A_m},\quad k=1,...,(n-1),n;\quad m=1,...,(n-1),n.$$

At that the system (2) will be written down in the following way:

$$F\left(A^j,P_0\right)=0. \tag{4}$$

Here A^j is a vector of solution which corresponds to the value of the conducting parameter $P=P_0$.

Let us change the conducting parameter for a small quantity $P_1=P_0+\Delta P$. At that the condition of equality of the vector to zero (4) will be violated. To restore the equality and to find solution of equation under the value of the conducting parameter P_1 let us precise the vector A^j and restore the condition of equality to zero (4):

$$F\left(A^j+\Delta A,P_1\right)=0. \tag{5}$$

If we expand the expression into the Taylor's' series and linearize the expanding, we will receive the following approximate formula:

$$F\left(A^j,P_1\right)+W\left(A^j\right)\Delta A=0. \tag{6}$$

It follows the correction to initial approximate answer

$$\Delta A=-W^{-1}\left(A^j\right)F\left(A^j,P_1\right). \tag{7}$$

After that we receive the improved answer as well:

$$A^{j+1}=A^j-W^{-1}\left(A^j\right)F\left(A^j,P_1\right). \tag{8}$$

Having repeat such operations, i.e. having carry out iteration process we will finally receive the problem solution with desired precision. The method is referred to as a Newton-Cantorovich method if Jacobian W for system on non linear equations is found for each iteration on précising solution. If Jacobi matrix is found only once under the first iteration and is used for all next iterations, the method is referred to as a modified method of Newton-Cantorovich.

As it seems from formula (8), the method of Newton-Cantorovich demands the converse of Jacobi matrix. It appears from this that the usage of the method is restricted due to absence of degeneracy of Jacobi matrix. Thus if Jacobi matrix is non degenerate, the method is applicable. It takes place if the system (5) is uniquely resolvable about the point of expansion into series relatively to A_J.

We start the problem solution from some initial point $P = P_0$, in which the solution is known. We move through the chain:

$$P_0,\ P_1,\ P_2,\ \ldots,\ P_j,\ P_{j+1},\ \ldots\ . \tag{9}$$

We forecast an approximate answer for each subsequent step of value of the conducting parameter by linear or second degree formula using the solutions of preceding steps [2]. At each step of movement through parameter we control increment of each coordinate of solution vector and if the increment of any coordinate is more than the step of conducting parameter, we change the coordinate into the conducting parameter and replace the coordinate turned into the conducting parameter in vector solution by the most conducting parameter.

In addition we automatically control the number of iterations on each step and either increase or reduce the step of movement by parameter taking into account the number of iterations necessary for ensuring solution for the concerned step with desired precision. Such procedure not only ensures condition for existence and uniqueness of solution, but also ensures movement with optimal speed to the objective and its achievement if the movement curve is not discontinuous.

This method was created half century ago with reference to computer technologies, when computers were rarities [1,2].

The methods of solution of non linear equations existed even at that time. Also convergence conditions of iteration processes were known, but because of lacking possibility for control of values and changes of parameters during the numerical solution of the problem, there were no ideas about conduct and leaded parameters and such problem was not considered at all. First the problem came up in the 60s of the last century [1]. When creating my method, we restricted its application area with the elastic systems, although for the first time we implemented the method of changing parameters by the method of Newton-Cantorovich for thin elastic shells.

We also never made a claim for authorship of the method of Newton-Cantorovich and moreover for the authorship of initial parameters method and other even lesser known method of solution of non linear equations that existed previously to us. But the main reason forced us to recall the early work, was regret that essence of method which we offered for today, for computer century is not understood by all. That's why only recently, in this century its application exceeded the bounds of non linear problems of the theory of thin elastic shells [5-9] and made it possible to review some traditional opinions.

In the past, and at present non linear boundary value problems are often solved by approximate analytical methods and methods of small parameter, Bubnov-Galerkin method, formal power series, methods of reduction to sequence of linear differential equations, Chapligin in 1905 was among the first to develop its variants. These approximate analytical methods are applied up to this day, though quite often under the global approaches to non linear problems the effects may be lost that are peculiar only to complicated non linear events.

As we have mentioned, small parameter method is also applied for this purpose, that has been used for researching of the thread tension and form under the action of centrifugal force of inertia when in use [10]. The following non linear differential equation therewith was received and used:

$$\left(1-\frac{1}{2}\varepsilon y^2\right)y''=-\varepsilon y\left(\lambda^2+y'^2\right);\ y(0)=0;\ y(1)=1. \tag{10}$$

For illustration of our opinion regarding the difference between numerical solution and solution of non linear problem by approximate analytical method we will use (10). The model contains only 2 parameters and comes to non linear problem with one unknown quantity, though my method is standardized, the program has been worked out for it and it makes no difference whether are hundreds or units of unknown quantities.

For solution of non linear equation the method of small parameter is applied in paper [10]. The small parameter is the parameter ε because in some experimental observations it was detected that this parameter is less than unity. But the question to what extent the observations are of generic character and how thoroughly the nature of the investigated event is reflected is not discussed in the paper and the question re-mains open. The question about the values of the second, geometrical parameter λ and the question whether the parameters of the problem are interdependent is also out of consideration. Particularly, whether is it possible to consider the second parameter ε to be small quantity under all values of parameter λ and to present solution of the equation (10) as an expansion in series by degree of small parameter ε taking into account smallness of the parameter? As it is shown in the work [10], as a result the sequence of linear solutions is received. Solution of this consecutive order begins from exact analytical definition of the null approximation. After that the first approximation is received since by means of the null approximation the equation for receiving the first approximation is précised. The equation can be also solved by the standard analytical method and the null approximation can be précised. However as long as the first and the next approximations do not insert more than 1% amendments into the null approximation, the conclusion is made in the work [10] that the process is converged rapidly and there is no necessity to precise the solution with the high approximations and the precision of the null approximation is sufficient [10].

All the said is logical on the ground of existing traditional linear mathematical thinking but in non linear problems the traditional thinking is not all powerful. The fact that the high approximations do not make essential contribution to the null approximation can be the evidence both of good and of bad convergence of the iteration processes. Moreover, if there is no the necessity to precise the solution, it means that non linear constituents should be neglected. As a result the problem turns to the linear one and small parameter method is not needed at all for its solution. There is no difference whether the parameter ε is large or small and more or less than unity. Let us solve the problem numerically and apply the method of Newton-Cantorovich in combination with the method of change of parameter [1,2].

Let us take into consideration the circumstance and take new indications and rewrite the non linear equation in the following way (if $y = y_1,\ y' = y_2$)

$$y_1' = y_2,\ y_2' = -\frac{\varepsilon y_1 \left(\lambda^2 + y_2^2\right)}{1 - 0.5\varepsilon y_1^2},\ y_1(0) = 0,\ y_1(1) = 0. \tag{11}$$

For numerical solution of the system of non linear equations, it is necessary to find the initial position of the fillet. After that using the intermediary solution corresponding to that initial position, the required final position of the fillet can be found. As such initial position let us take the weightless cord position which is stretched straight between the points of the considered piece of the fillet that under the conditions of the problem defines the operative condition of the fillet. The formulas for this intermediary solution follow from the correlations and can be presented as

$$\varepsilon = (8,2),\ y_1(0) = 0,\ y_2(0) = 1 + A(8,1),\quad F\left(A(8,1), A(8,2)\right) \equiv y_1(1) - 1 = 0. \tag{12}$$

At the initial position when the fillet has the form of stretched string, the following values of parameters $A(8,1) = A(8,2) = 0$ correspond to solution. The solution of the system (11) therewith looks like: $y_1(x) = x,\ y_2(x) = 1$.

The results of the calculations are presented in the table for the whole values of the geometrical parameter λ equaled to 1, 2, 4, 8. This parameter was of constant value and was not changed in the process on computation. What about two another parameters of the problem $A(8,1)$ and $A(8,2)$, their solutions at the beginning of the calculation process were equaled to zero and changed in the process of calculation and restrictions to their change were not applied. In so doing continuation of calculation was unimpeded and calculation stopped forcible at any fixed value of geometrical parameter λ if the received results could be considered sufficient.

Table

λ	A(8,1)	A(8,2)	$y_1(0)$	$y_1(0.25)$	$y_1(0.5)$	$y_1(0.75)$	$y_1(1)$	$y_2(0)$	$y_2(0.5)$	$y_2(1)$
1	0.7677	1.015	0.000	0.4234	0.7517	0.9435	1.000	1.768	1.048	-0.3393
1	1.025	1.080	0.000	0.4789	0.9926	0.9926	1.000	2.025	1.025	-0.2825
1	1.251	1.096*	0.000	0.5264	0.8823	1.032	1.000	2.251	0.9979	-0.4902
1	2.096	1.009	0.000	0.6974	1.082	1.169	1.000	3.096	0.8792	-1.265
2	0.0966	0.1063	0.000	0.2726	0.5398	0.7843	1.000	1.097	0.8150	0.8150
2	0.2255	0.2219	0.000	0.3025	0.5824	0.8199	1.000	1.225	1.047	0.5904
2	1.675	0.5922*	0.000	0.6253	1.038	1.163	1.000	2.675	1.085	-1.236
2	1.858	0.5893	0.000	0.6644	1.088	1.200	1.000	2.858	1.074	-1.433
4	0.1825	0.0565	0.000	0.2926	0.5674	0.8079	1.000	1.183	1.104	0.6558
4	2.620	0.2462*	0.000	0.8408	1.358	1.414	1.000	3.620	1.180	-2.526
4	3.071	0.2431	0.000	0.9386	1.490	1.490	1.000	4.071	1.171	-3.022
4	3.318	0.2401	0.000	0.9918	1.560	1.563	1.000	4.318	1.164	-3.279
8	0.2577	0.0201	0.000	0.3101	0.5949	0.8312	1.000	1.258	1.056	0.5200
8	0.6442	0.0408	0.000	0.3995	0.7324	0.9439	1.000	1.644	1.119	-0.1071
8	4.188	0.0853*	0.000	1.197	1.874	1.802	1.000	5.188	1.276	-4.407
8	4.639	0.0850	0.000	1.296	2.014	1.904	1.000	5.639	1.275	-4.881

In the 1-st column of the table the values of parameter λ are presented. In the 2-nd and 3-rd columns values of parameters $A(8,1)$ and $A(8,2)$ are presented. The former, $A(8,1)$, as it follows from ratio for the angle (12) is the angle of rotation of tangent to the axis of fillet in the anterior section $x = 0$ relatively to initial line.

Both in this and any other point of the fillet the difference between the angle of slope to the horizon between the tangent to the axis of the fillet and initial line in the beginning of the process is equaled to unity.

As may be seen from the table, the 1-st parameter $A(8,1)$, in the counting process in all considered cases is changing monotonically as against to the 2-nd parameter $A(8,2) = \varepsilon$. It has maximal value in all cases that is marked with asterisk in the table. This maximal value of parameter ε under the small quantities of λ is more than unity. If the geometrical parameter λ increases, it gradually subsides and becomes less than unity. That's why it is wrongful a priori to affirm that this parameter is always small and less than unity.

From the analysis of the table data transpires that if this parameter is taken as a conducting one about the points of maximal value of parameter $A(8,2)$, the condition for existence and uniqueness of solution is violated, Jacobian of non linear system of equations amounts to zero and using of any method of solution of systems of non linear equations, including method of Newton-Cantorovich becomes wrongful. It may appear that since the parameter $A(8,1)$ is changing monotonely it can always be taken as conducting one and the full picture of possible solutions can be received. But this impression is deceptive as in so doing the whole areas of existence of possible solutions can be lost. In such areas the interdependency of the parameters can be complicated and even folded that has many extreme and inflexion points. Passing through these points needs the compulsory control of parameters and in case of necessity change over conducting and leaded parameters [2].

Interchanges and angles of slope of tangents in the characteristic points of fillet are listed in the table. For perception of quantitative and qualitative significance and meaning of the design values it is reasonable to remember the initial position of the fillet that is outlined by the straight line sloping at the angle 45° to the horizon.

It is clear from the table that the definite condition is met in all points of the deformed fillet - the points are located above the appropriate points of the initial curve, i.e. axis of the deformed fillet is convex curve and in the finite points this condition coincides with conditions (12): $y_1(x) - x \geq 0$, $y_1(0) = 0$, $y_1(1) = 1$. If $\lambda = 8$ the picture is qualitatively close to the results received in the paper [10] by method of small parameter.

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