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ON THE UNSTEADY MOTION OF A VISCOUS HYDROMAGNETIC FLUID CONTAINED BETWEEN ROTATING COAXIAL CYLINDERS OF FINITE LENGTH

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Abstract. The problem of unsteady rotation motion of electrically conducting viscous incompressible fluid, contained within two axially concentric cylinders of finite length in the presence of an axial symmetric magnetic field of constant strength, has been solved exactly using finite Hankel transform in combination with a technique presented in this paper. This paper presents a complete of the problem under consideration, which has been of interest for many years; moreover the Pneuman-Lykoudis solution in Magnetohydrodynamics and Childyat solution in hydrodynamics appears as a special case of this study. The analysis shows that the disturbance in the fluid disappear by increasing the magnetic field.

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In recent years, the study of electrically conducting fluids has received much attention in areas such as rocket flight and high-speed reentry missiles. It is known that the motion of a conducting fluid in a magnetic field induces electric currents in the fluid, thereby modifying the field; at the same time, the flow in the magnetic field produces mechanical forces which in turn modify the motion.

Recently, C. D. Ghildyal has presented a solution of the problem in hydrodynamics concerning the unsteady motion of a viscous, rotating fluid contained between two infinitely long coaxial cylinders. However, the motion of an electrically conducting fluid contained between two coaxial rotating cylinders in the presence of magnetic field becomes much more complicated. S. Chandrasekhar has discussed the rotational and thermal instability of a viscous, rotating, electrically conducting fluid within two infinitely long coaxial cylinders in presence of a magnetic filed. In this paper, we are directly interested in the deformation of the fluid in the presence of a magnetic field; hence, we are searching for the velocity distribution in a magnetic field. That is the fundamental object of this study. After the fluid field is determined, we can study the electromagnetic field from the knowledge of the boundary and initial condition imposed upon the field. Not that a wide range of solutions compatible with these conditions are possible [1, 2, 4-7]. It is convenient for the study of this problem to introduce a cylindrical system. Let the length of the cylinders be $2L$, so that the origin of the system is located at the middle point on axis of cylinders. We assume that the velocity field is given by

$$\vec{V} = \vec{V}(\nu, u, w), \tag{1}$$

where ν, u, w are the components of the velocity in the radial, circumferential and axial directions, respectively. Since the motion of the fluid is primarily in the circumferential direct-ions, the radial velocity ν in the flow may be neglected on comparison with the circumferential velocity u. However, the assumption that the axial velocity w is zero is true only if the cylinde-rs are infinitely long. If the cylinders are finite in length, the axial velocity is not zero. Therefore, the "end effects" should be considered. Pneman and Lykoudis [8] showed that the "end effects" will be confined to regions very close to the bounding planes and the axial velocity may be safely neglected in the region midway between these planes.

Furthermore, we impose a magnetic field of constant strength. H_0, in axial direction and we assume that there are perturbations h_r, h_θ and h_z in the magnetic field and E_r, E_θ, E_z in the electrical field. Note that it can be assumed that $h_r \ll h_\theta$ and $h_r \ll h_z$, since the motion occurs in the θ direction. Therefore

$$\vec{H} = \vec{H}(0, h_0, H_0 + h_z). \tag{2}$$

Similarly,

$$\vec{E} = \vec{E}(0, E_\theta, E_z). \tag{3}$$

It has been assumed throughout this paper that physical and electromagnetic properties of the flow are known. Since the flow is incompressible with constant properties, the energy equation and characteristic equation may be omitted; therefore, the problem is reduced to the solution of the following system of equations, namely: Continuity:

$$\Delta \cdot \vec{V} = 0. \tag{4}$$

Momentum:

$$\frac{\partial \vec{V}}{\partial t} + (\nabla \cdot \vec{V})\vec{V} + \frac{1}{\rho}\nabla p - \frac{\mu}{\rho}\nabla p - \frac{\mu}{\rho}\nabla^2 \vec{V} - \frac{\mu_e}{\rho}(\vec{J} \times \vec{M}) = 0. \tag{5}$$

Maxwell equations:

$$\nabla \cdot \vec{D} - \rho_e, \tag{6}$$

$$\nabla \cdot \vec{H} = 0, \tag{7}$$

$$\nabla \times \vec{E} + \mu_e \frac{\partial \vec{H}}{\partial t} = 0, \tag{8}$$

$$\nabla \times \vec{H} - \vec{J} - \frac{\partial \vec{D}}{\partial t} = 0, \tag{9}$$

$$\nabla \cdot \vec{J} + \frac{\partial \rho_e}{\partial t} = 0. \tag{10}$$

Here are: $\vec{V}$ – velocity of the flow (not perturbed), ρ – density of the flow, p – pressure, μ – dynamical viscosity, $\vec{J}$ – current density, $\vec{H}$ – strength of magnetic field, μ_e – magnetic permeability, $\vec{E}$ – strength of electric field, ρ_e – charge density, t – time. Note that

$$\vec{B} = \mu_e \vec{H}, \tag{11}$$

$$\vec{D} = \varepsilon_e \vec{E}, \tag{12}$$

$$\vec{J} = \sigma(\vec{E} + \mu_e \vec{q} \times \vec{H}), \tag{13}$$

where $\vec{B}$ is magnetic induction, $\vec{D}$ is displacement current, ε_e is dielectric constant, σ is electric conductivity. The system of eq. (4), (5), (6), (7), (8), (9), (10) is subjected to the conditions imposed upon the hydrodynamic field and those imposed upon the electromagnetic field. Assuming that both cylinders are rotating in the same direction, the no-slip condition at the walls of the cylinders requires that

$$u(a, \theta, z, t) = \omega_1 a, \tag{14}$$

$$u(b, \theta, z, t) = \omega_2 b, \tag{15}$$

$$u(r, \theta \pm L, t) = \omega_e r, \tag{16}$$

where ω_1 and ω_2 are angular velocities of the cylinders, a is the radius of the inner cylinder, b is the radius of the outer cylinder, and ω_e is angular velocity of the bounding end planes.

Moreover, the initial condition, for $t = 0$, is prescribed, namely,

$$u(r, \theta, z, 0) = f(r, z), \tag{17}$$

where $f(r, z)$ is a regular function everywhere in the domain under consideration.

In addition to these conditions, we have the conditions imposed by electromagnetic field which require that on the surface of the discontinuity, such as $r = a$ and $r = b$, the normal and tangential components of the magnetic induction and electric field suffer a discontinuity, which is equal for the magnetic field to the components of the surface current density at the right angle to the field, and for the electric field to the components of the surface charge density perpe- ndicuar to the electric field. Note that the surface current density is measured in amper per meter, and surface charge density is Coulombs per square meter. However, in accordance with the assumption that the bounding end planes are of insulating material, we have that the normal and tangential components of the magnetic induction and electric field are continuous on those planes.

The main characteristic of the field equations (6), (7), (8) and (10) is that electric field depends upon the magnetic field through the time variation of $\vec{H}$; and the magnetic field depends on the electric field through the time variation of $\vec{D}$. Therefore, the initial condition in the electromagnetic field has to be introduced.

Evidently, the time derivative of $\vec{D}$ acts as a source for $\vec{H}$, and the derivative of $\vec{H}$ acts as a source for $\vec{E}$. Hence, coupling between the electric field and the magnetic field becomes bilateral. Note that the hydrodynamic field is coupled with the electromagnetic field through the electrobody force. At the first glance, the problem from the mathematical point of view represe-nts a mixed boundary value problem of a very complicated nature. That is true. However, the problem can be solved using the method of the operational calculus outlined in this paper. From the continuity eq. (4), since circumferential symmetry is assumed, it follows that

$$u = u(r, z, t). \tag{18}$$

Then (5) and (17) lead to following equations in scalar form:

$$-\rho\frac{u^2}{r} - F_r + \frac{\partial p}{\partial r} = 0, \tag{19}$$

$$\rho\frac{\partial u}{\partial t} - \mu\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\Big) - F_\theta = 0, \tag{20}$$

$$F_z - \frac{\partial p}{\partial z} = 0, \tag{21}$$

where F_r, F_θ, F_z are coordinates of the electrobody force in r, θ and z direction, respectively. Then

$$\vec{F} = \vec{J} \times \vec{B}, \tag{22}$$

where $\vec{J}$ and $\vec{B}$ are prescribed by (11) and (13). However, it can be shown that h_z is zero everywhere in the field. The proof is simple. By circumferential symmetry (7) becomes

$$\frac{\partial h_z}{\partial z} = 0, \tag{23}$$

or

$$h_z = h_z(r,t). \tag{24}$$

Moreover, on the boundary $z = \pm L$ we have that the magnetic field is constant, namely $\vec{H} = H_0\vec{e}_z$. Hence, $h_z = 0$ at the planes $z = \pm L$, and therefore is zero everywhere. This completes the proof. Hereafter, $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_z$ denote the unit vectors in r, θ and z – direction.

The problem will be solved if.

$$\rho\frac{\partial u}{\partial t} - \mu\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r} + \frac{\partial^2 u}{\partial z^2}\Big) + \sigma\mu_e^2 H_0^2 u = 0, \tag{25}$$

subjected to the conditions (14), (15), (16) and (17) is solved.

For such purposes, denote

$$\frac{\mu}{\rho} = \nu; \quad \frac{\sigma\mu_e^2 H_0^2}{\mu} = \Omega^2. \tag{26}$$

Then (25) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\Big(\frac{1}{r^2} + \Omega^2\Big)u + \frac{\partial^2 u}{\partial z^2} - \frac{1}{\nu}\frac{\partial u}{\partial t} = 0. \tag{27}$$

Evidently, if $\Omega = 0$ and in addition $L \to \infty$ then the solution is reduced to the simple solution, recently given in hydrodynamics for infinite long cylinders. This will be later on evidently, from resulting velocity field.

Written in operator form, the boundary value problem, under consideration, can be written as

$$\left.\begin{array}{r} D\{u(r,z,t)\} = 0, \\ u(r,z,0) = f(r,z), \\ \left.\begin{array}{r} u(a,z,t) = \omega_1 a, \\ u(b,z,t) = \omega_2 b, \\ u(r,\pm L,t) = \omega_e r, \end{array}\right\} \quad t > 0 \end{array}\right\}. \tag{28}$$

In conclusion we will get:

$$\begin{aligned}u(r,z,t)=\sum_{m}^{\infty}\sum_{n}^{\infty}\Big[A_{mn}+\frac{\pi^2}{2}\frac{\alpha_m^2 J_1(\alpha_m b)}{J_1^2(\alpha_m a)-J_1^2(\alpha_m b)}M_{mn}\Big]\times\\ \times B_1(\alpha_m r)\cos A_n z e^{-\nu(k_m^2+A_n^2)t}+\\ +\frac{\pi^2}{2}\sum_{m}^{\infty}\frac{\pi^2}{2}\frac{\alpha_m^2 J_1(\alpha_m b)}{J_1^2(\alpha_m a)-J_1^2(\alpha_m b)}\times\\ \times B_1(\alpha_m r)\Big[\Big(G_2(\alpha_m)+\\ +\frac{A^*(\alpha_m)}{k_m^2}\Big)\frac{\cosh k_m z}{\cosh k_m L}-\frac{A^*(\alpha_m)}{k_m^2}\Big]=T^*+S^*,\end{aligned}$$

where

$$\begin{aligned}A_{mn}=&\frac{2}{L}\frac{\int_a^b\int_{-L}^{+L} rf(r,z)B_1(\alpha_m r)\cos A_n z\,dr\,dz}{b^2B_2^2(\alpha_m b)-a^2B_2^2(\alpha_m a)},\\ A_n=&\frac{2n+1}{2L}\pi\quad(n=0,1,2,\dots),\\ M_{mn}=&\frac{2\sin A_nL}{A_nL}\Big\{\frac{A^*(\alpha_m)}{k_m^2}-\Big[G_2(\alpha_m)+\\ &+\frac{A^*(\alpha)m)}{k_m^2}\Big]\frac{A_n^2}{k_m^2+A_n^2}\Big\},\quad k_m^2=\alpha_m^2+\Omega^2,\\ G_2(\beta)=&\frac{\omega_e}{\beta}\big[b^2(J_2(\beta b)Y_1(\beta a)-Y_2(\beta b)J_1(\beta a))-\\ &-a^2J_2(\beta a)Y_1(\beta a)-Y_2(\beta a)J_1(\beta a)\big]\\ B_1(\beta r)=&J_1(\beta r)Y_1(\beta a)-Y_1(\beta r)J_1(\beta a).\end{aligned}$$

This solution for $f(r,z)$ is obtained by means of Hankel Transform in the form of the Bessel functions the first and second kind $J_\tau(\ )$, $Y_\tau(\ )$, of the order $\tau=1,2$, namely

$$\begin{aligned}f(r,z)=&\frac{\pi^2}{2}\sum_{\beta}\frac{\beta^2J_1(\beta b)}{J_1^2(\beta a)-J_1^2(\beta b)}\times\\ &\times\Big\{[G_2(\beta)-G_1(\beta)]\frac{\cosh\sqrt{\beta^2+\Omega^2}\,z}{\cosh\sqrt{\beta^2+\Omega^2}\,L}+G_1(\beta)\Big\}B_1(\beta r),\end{aligned}$$

where

$$G_1(\beta)=\frac{2\omega_2 b}{\pi(\beta^2+\Omega^2)}\frac{J_1(\beta a)}{J_1(\beta b)}.$$

The problem of unsteady rotational motion of electrically conducting, viscous, incompressible fluid contained within two axially concentric cylinders of finite length in the presence of an axial symmetric magnetic field, has been studied and solved by methods of methods of the operational calculus. The solution for the velocity field is obtained in an exact from, since the corresponding partial differential equations are solved exactly.

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