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ON THE IMPACT OF TWO AXIALLY SYMMETRIC SPATIAL STREAMS

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Abstract. On the construction of solutions of the spatial axially symmetric stationary problem with partially unknown boundaries problems of the theory of jet flows. In the work we present a general mathematical method of solutions os spatial axially symmetric with partially unknown boundaries problems of the theory of jet flows.

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In the 40s of the past century, the researches engaged in the theory of filtration discovered that the stream theory can be applied to cumulative charges whose action is based on the peculiarity that explosion is directed in one and the same side. Detonation wave distributes over an explosive substance with rate of several km/sec, and behind the wave front there arises pressure of order 10^5 atmospheres. Metallic cover behaves itself as a perfect liquid ([1-3]). The symmetry axis is assumed to be the x-axis. Of an infinite set of half-planes we choose arbitrarily one which passes through the symmetry axis the moving liquid on which occupies certain simply connected domain $S(z)$, where $z = x + iy$, a part of its boundary is unknown and due to be defined. A surface is said to be a surface if it is formed by the rotation of some curve near the axis. The velocity potentials $\varphi(x, y)$ and the flow function $\psi(x.y)$ are the functions of only cylindrical coordinates x and y, where y is the distance from the x-axis. Owing to the axial symmetry, it suffices, as is said above, to study the flow in an arbitrarily chosen meridional half-plane with the right system of coordinates (x, y). As is known, the above-mentioned functions satisfy the following equations ([1-3]):

$$\frac{\partial\varphi}{\partial x} = \frac{1}{y}\frac{\partial\psi}{\partial y} = v_x; \quad \frac{\partial\varphi}{\partial y} = -\frac{1}{y}\frac{\partial\psi}{\partial x} = v_y, \tag{1}$$

where v_x and v_y are the velocity projections onto the axes x and y. It follows from (1) that

$$\frac{\partial}{\partial x}\left(y\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(y\frac{\partial\varphi}{\partial y}\right) = 0; \quad \frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial\psi}{\partial y}\right) = 0. \tag{2}$$

The problem can be assumed to be solved if any of two functions $\varphi(x, y)$ or $\psi(x, y)$ is known. For their determination, we have, besides the equations (2), the following boundary conditions ([1-3]):

(a) normal velocity on an unknown free surface and on the body surface is equal to zero, $\frac{\partial\varphi}{\partial n} = 0$, where n is the normal, directed into the liquid;

(b) the function ψ on the body surface and on the free surface is constant, $\psi(x,y) = const$;

(c) the condition $\psi(x,y) = const$ is equivalent to the condition $\frac{\partial\varphi}{\partial n} = 0$;

(d) along the symmetry axis x we have $\left(\frac{\partial\varphi}{\partial y}\right)_{y\to 0} = 0$, $\left(\frac{\partial\psi}{\partial y}\right)_{y\to 0} = 0$, $\left(\frac{\partial\psi}{\partial x}\right)_{y\to 0} = 0$, $\left(\frac{1}{y}\frac{\partial\varphi}{\partial y}\right)_{y\to 0} = \frac{\partial^2\varphi}{\partial y^2}$, $\left(\frac{1}{y}\frac{\partial\psi}{\partial y}\right)_{y\to 0} = \frac{\partial^2\psi}{\partial y^2}$.

The system (2) can be written as follows:

$$\frac{\partial^2\varphi}{\partial x^2} + 4\alpha\,\frac{\partial^2\varphi}{\partial x^2} + 4\,\frac{\partial\varphi}{\partial x} = 0, \qquad \frac{\partial^2\psi}{\partial x^2} + 4\alpha\,\frac{\partial^2\psi}{\partial x^2} = 0, \ \ \alpha = y^2. \tag{2_1}$$

The half-plane $\mathrm{Im}(q) \geq 0$ (or $\mathrm{Im}(q) < 0$) of the plane $q = q_1 + iq_2$ we map conformally onto the domain $S(z)$, where $z(q) = z(q_1,q_1) + iy(q_1,q_2)$ is a part of the boundary $S(\ell)$ of the domain $S(a)$ which is unknown beforehand and due to be defined. The system (1) on the plane $q = q_1 + iq_2$ takes the form ([1-3])

$$\begin{aligned} &\frac{\partial}{\partial q_1}\left(y(q_1,q_2)\,\frac{\partial\varphi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(y(q_1,q_2)\,\frac{\partial\varphi}{\partial q_2}\right) = 0,\\ &\frac{\partial}{\partial q_1}\left(\frac{1}{y(q_1,q_2)}\,\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{1}{y(q_1,q_2)}\,\frac{\partial\psi}{\partial q_2}\right) = 0. \end{aligned} \tag{3}$$

For three analytic functions we introduce the notations:

$$\begin{gathered} z(q) = x(q_1,q_2) + iy(q_1,q_2), \ \ \omega_0(q) = \varphi_0(q_1,q_2) + i\psi_0(q_1,q_2),\\ w_0(q) = \frac{\omega_0'(q)}{z'(q)}\,, \end{gathered} \tag{4}$$

which map conformally the half-plane $\mathrm{Im}(q) \geq 0$, respectively, onto the domain $S(z)$ of liquid motion, onto the domain of a complex potential $S(\omega_0(q))$ and onto the domain of a complex velocity $S(w_0(q))$. Below, the functions $z(q), \omega_0(q)$ and $w_0(q)$ will be assumed as defined.

We seek for a solution (3) in the form

$$\begin{aligned} \varphi(q_1,q_2) &= y^{-1/2}(q_1,q_2)[\varphi_2^*(q_1,q_2) + \varphi_2(q_1,q_2)],\\ \psi(q_1,q_2) &= y^{1/2}(q_1,q_2)[\psi_2^*(q_1,q_2) + \psi_2(q_1,q_2)]. \end{aligned} \tag{5}$$

Substituting (5) respectively into (3), we obtain

$$\Delta(\varphi_2^* + \varphi_2) = -\frac{1}{4}\rho_1(\varphi_2^* + \varphi_2), \ \ \Delta(\psi_2^* + \psi_2) = \frac{3}{4}\rho_1(\psi_2^* + \psi_2), \tag{6}$$

where

$$\rho_1 = \left(\frac{1}{y}\,\frac{\partial y}{\partial q_1}\right) + \left(\frac{1}{y}\,\frac{\partial y}{\partial q_2}\right). \tag{7}$$

Applying the Green's formula to the upper half-plane, we can get from (6) the following Fredfolm integral equation of second kind:

$$\varphi_2(q_1,q_2) + \frac{1}{4}\iint_{\mathrm{Im}(q)\geq 0} G(q_1,q_2;x_1,y_1)\rho_1(x_1,y_1)\varphi_2(x_1,y_1)dx_1dy_1 =$$

$$= f_1(q_1, q_2), \tag{8}$$

where

$$f_1(q_1,q_2) = -\varphi_2^*(q_1,q_2)$$
$$-\frac{1}{4}\iint_{\mathrm{Im}(q)\geq 0} G(q_1,q_2;x_1,y_1)\rho_1(x_1,y_1)\varphi_2^*(x_1,y_1)dx_1dy_1, \tag{9}$$

$$G(q_1,q_2;x_1,y_1) = \frac{1}{4\pi}\ln\frac{(q_1-x_1)^2+(q_2+y_1)^2}{(q_1-x_1)^2+(q_2-y_1)^2} \tag{10}$$

is the Green's function for $\mathrm{Im}(q) \geq 0$.

A solution of the integral equation (8) will be sought by using the method of successive approximations in the form of the following series:

$$\varphi_1(q_1,q_2) = \sum_{n=0}^{\infty}\lambda^n\varphi_{2(n)}(q_1,q_2). \tag{11}$$

Substituting the series (11) into the equation (8) we obtain:

$$\varphi_{2(0)}(q_1,q_2) = f_1(q_1,q_2), \tag{12}$$

. . . .

$$\varphi_{2(n)}(q_1,q_2) = \iint\limits_{\mathrm{Im}(q)\geq 0} G(q_1,q_2;x_1,y_1)\rho_1(x_1,y_1)\varphi_{2(n-1)}(x_1,y_1)dx_1dy_1, \tag{13}$$

. . . .

Assume that two streams A_1 and A_3 collide and scatter in the form of the streams A_2 and A_4 under the condition that pressure and moduli of velocity v on every free surface are the same ([1-3]).

We introduce complex velocity of the flow $\zeta = \frac{d\omega_0}{dz} = ve^{-i\theta}$. It can easily be verified that when passing round the boundaries of the domain in the clockwise direction (A_1, A_2, A_3, A_4), the angle θ varies from zero to 2π, however, the argument of the function $\zeta = \frac{d\omega_0}{dz}$ varies from zero to -2π because on the boundary of the flow domain the maximal value is $v = 1$. To that domain on the plane ζ there corresponds the unit circle $|\zeta| < 1$. The width of the streams A_1, A_2, A_3, A_4 at infinity we denote respectively by a_1, a_2, a_3, a_4. When passing through the points $A_1(\zeta = 1)$, $A_2(\zeta = e^{-i\theta_2})$, $A_3(\zeta = -1)$, $A_4(\zeta = e^{i\theta_2})$, the imaginary part of the complex potential undergoes jumps equal to $a_1, -a_2, a_3, -a_4$, and hence the function $w(\zeta)$ at those points has logarithmic singularities. Under analytical extension to the whole plane of variable ζ, the function $w(\zeta)$ has no new singularities, and we can write ([1-3]) (see Fig. 1)

$$\omega_0(\zeta) = \frac{1}{\pi}[a_1\ln(\zeta-1) + a_3\ln(\zeta+1) - a_2\ln(\zeta-e^{-i\theta_2}) - a_2\ln(\zeta-e^{i\theta_2})]. \tag{14}$$

The liquid amount brought by the streams A_1 and A_3 is equal to that taken away by the streams A_2 and A_4. Then $a_1 + a_3 = 2a_2$. When this condition is fulfilled, the function $\omega_0(\zeta)$ is regular for $\zeta = 0$ and $\zeta = \infty$. The hodograph of the flow under consideration is a circle.

The function $z(\zeta)$ has the form

$$\begin{aligned} z(\zeta) = \frac{1}{\pi}[a_1 \ln(1-\zeta) - a_3 \ln(1+\zeta) \\ - a_2 e^{i\theta_2} \ln(1-\zeta e^{i\theta_2}) - a_2 e^{-i\theta_2} \ln(1-\zeta e^{i\theta_2})]. \end{aligned} \tag{15}$$

If $\theta_3 = \pi$, and the middle lines of the streams A_1 and A_3 coincide, then the flow is symmetric with respect to the x-axis, i.e., x is the flow line.

In a particular case, for the determination of unknown parameters we have the following system of equations ([1-3]):

$$\theta_2 + \theta_4 = 2\pi, \;\; a_2 = a_4; \;\; a_1 + a_3 = 2a_2, \;\; a_1 - a_3 = 2a_2 \cos\theta_2, \;\; \cos\theta_1 = \frac{a_1 - a_3}{a_1 + a_3}.$$

From (14) and (15) we have

$$\begin{aligned} \frac{d\omega_0}{d\zeta} &= \frac{8a_1 a_3 \zeta}{\pi(a_1+a_3)(\zeta^2-1)(\zeta - e^{i\theta_2})(\zeta - e^{-i\theta_2})}, \\ \frac{dz}{d\zeta} &= \frac{1}{\pi}[a_1 \ln(1-\zeta) - a_3 \ln(1+\zeta) \\ &\quad - a_2 e^{i\theta_2} \ln(1-\zeta e^{i\theta_2}) - a_2 e^{-i\theta_2} \ln(1-\zeta e^{i\theta_2})]. \end{aligned}$$

The function $q = q_1 + iq_2$ is connected with the function ζ by the formula

$$\zeta = \frac{i-q}{i+q}, \quad |\zeta| \leq 1, \quad \mathrm{Im}(q) \geq 0.$$

The solution (8) can be considered as the first approximation, and the subsequent approximations can be obtained by using formula (13).

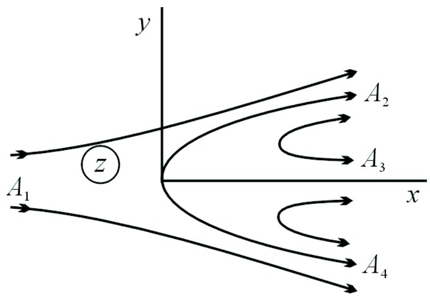

Fig. 1

R E F E R E N C E S

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