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THE LAGRANGE PRINCIPLE OF TAKING RESTRICTIONS OFF IN QUASILINEAR CONTROL SYSTEMS WITH DELAYS manana

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Abstract. Using the Lagrange Principle of taking restrictions off the necessary conditions of optimality for quasilinear systems with mixed restrictions and delays are given. As against before executed works necessary conditions of optimality in case of continuous initial function, that is most natural for processes with delays, are proved.

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We consider the following problem

$$I = \int\limits_{t_0}^{t_1} f^0\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right)\right) dt \to inf \tag{1}$$

under the restrictions

$$\dot{x}(t) = f\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right)\right), \tag{2}$$

$$g\left(x\left(t\right), u\left(t\right)\right) \leq 0, t \in [t_0, t_1], \tag{3}$$

$$x\left(t_1\right) = x_1, \tag{4}$$

$$x\left(t\right) = \hat{\varphi}\left(t\right), t \in [t_0 - \Theta, t_0], \tag{5}$$

where $\hat{\varphi}$ is fixed vector function, $t_0, t_1, \Theta > 0$ are fixed numbers, the scalar function f^0 and the vector functions $f \in R^n, g \in R^m$ are continuously differentiated with respect to all their arguments. In different from general problems, which are considered in [1], [2], the problem (1) - (5) includes mixed restrictions (3).We consider a case, when the restrictions (3) fulfilled the conditions of generality: for any (x, u), satisfying (3), the system of vectors $grad_u g^j(x, u), j \in J(x, u)$ is linearly independent. Here by $J(x, u)$ we denote the set of indices $j \in \{1, 2, \cdots, m\}$, for which $g^j(x, u) = 0$. Let vector function $\hat{\varphi(t)}$ is continuous on $[t_0 - \Theta, t_0]$, the vector function $x(t)$ is absolutely continuous, the vector function $u(t)$ is integrable on $[t_0, t_1]$, i.e. $\hat{\varphi} \in C[t_0 - \Theta, t_0]$, $x(t) \in W^n_{1,1}[t_0, t_1]$, $u(t) \in L^r_1[t_0, t_1]$, the conditions (2) - (3) are fulfilled for almost all $t \in [t_0, t_1]$, the functions $f^0, f \in R^n, g \in R^m$ are linear to control parameters and satisfying condition of "convexity" (see [3]), then the following theorem states necessary conditions of optimality for problem (1) - (5).

Theorem 1. *Let $(x(t), u(t))$ is e solution of the problem (1) - (5). Then there exist multipliers $\psi_0 \geq 0$, $\psi(t) \in W^n_{1,1}[t_0, t_1]$ and $\mu(t) \in L^m_\infty[t_0, t_1]$, such that, almost everywhere on $[t_0, t_1]$ the following conditions are fulfilled*

$$\mu_j\left(t\right) \geq 0, \tag{6}$$

$$\mu_j\left(t\right) g^j\left(x\left(t\right), u\left(t\right)\right) = 0, j = \overline{1, m}, \tag{7}$$

$$H\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right), \psi_0, \psi\left(t\right)\right)$$
$$= \min_{u \in \{u | g(x(t), u) \leq 0\}} H\left(x\left(t\right), x\left(t-\Theta\right) u, \psi_0, \psi\left(t\right)\right), \tag{8}$$

$$\frac{d\psi}{dt} = \frac{\partial \Re\left(x\left(t\right), x\left(t-\Theta\right), u(t), \psi_0, \psi\left(t\right), \mu\left(t\right)\right)}{\partial x(t)}$$

$$+ \frac{\partial \Re\left(x\left(t+\Theta\right), x\left(t\right), u(t+\Theta), \psi_0, \psi\left(t+\Theta\right), \mu\left(t+\Theta\right)\right)}{\partial x(t-\Theta)}, t \in [t_0, t_1-\Theta], \tag{9}$$

$$\frac{d\psi}{dt} = \frac{\partial \Re\left(x\left(t\right), x\left(t-\Theta\right), u(t), \psi_0, \psi\left(t\right), \mu\left(t\right)\right)}{\partial x(t)}, t \in [t_1-\Theta, t_1], \tag{10}$$

$$\frac{\partial \Re\left(x\left(t\right), x\left(t-\Theta\right), u(t), \psi_0, \psi\left(t\right), \mu\left(t\right)\right)}{\partial u(t)} = 0, \tag{11}$$

and

$$(\psi_0, \psi(t)) \neq (0, 0), \tag{12}$$

where

$$H\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right), \psi_0, \psi\left(t\right)\right)$$
$$\equiv \psi_0 f^0\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right)\right) - \sum_{i=1}^{n} \psi_i\left(t\right) f^i\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right)\right),$$
$$\Re\left(x\left(t\right), x\left(t-\Theta\right), u(t), \psi_0, \psi\left(t\right), \mu\left(t\right)\right)$$
$$\equiv H\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right), \psi_0, \psi\left(t\right)\right) + \mu\left(t\right) g\left(x\left(t\right), u\left(t\right)\right).$$

Proof. Let's enter continuous functions $z(t)$and we shall present a problem $(1)-(5)$ in the form

$$\int_{t_0}^{t_{0+\Theta}} f^0\left(x\left(t\right), z\left(t-\Theta\right), u\left(t\right)\right) dt + \int_{t_{0+\Theta}}^{t_1} f^0\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right)\right) dt \rightarrow inf \tag{13}$$

under the restrictions:

$$\dot{x}(t) = f\left(x\left(t\right), z\left(t-\Theta\right), u\left(t\right)\right), t \in [t_0, t_0+\Theta], \tag{14}$$

$$\dot{x}(t) = f\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right)\right), t \in [t_0+\Theta, t_1], \tag{15}$$

$$g\left(x\left(t\right), u\left(t\right)\right) \leq 0, t \in [t_0, t_1], \tag{16}$$

$$x\left(t_1\right) = x_1, \tag{17}$$

$$z\left(t\right) = \hat{\varphi}\left(t\right), t \in [t_0-\Theta, t_0]. \tag{18}$$

Clearly, that the problem (13) – (18) is a special case of an extreme problem:

$$f_0(w) \to inf \mid F(w) = 0, f_i(w) \leq 0, (i = \overline{1,n}), w \in W, \tag{19}$$

where W is Banach-space. Indeed,

$$W = C^n[t_0-\Theta, t_0] \times W_{1,1}^n[t_0, t_1] \times L_1^r[t_0, t_1] \times L_2^m[t_0, t_1]$$

$$Y = C^n[t_0-\Theta, t_0] \times L_1^{n+m}[t_0, t_1],$$

$$w = (z, x, u, y),$$

$$f_0(w) = \int\limits_{t_0}^{t_0+\Theta} f^0\left(x\left(t\right), z\left(t-\Theta\right), u\left(t\right)\right) dt + \int\limits_{t_0+\Theta}^{t_1} f^0\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right)\right) dt,$$

$$f_i(w) = x^i(t_1) - x_1^i, i = \overline{1,n},$$

$$F(w) = \begin{cases} z(t) - \hat{\varphi}(t), t \in [t_0-\Theta, t_0], \\ \dot{x}(t) - f\left(x\left(t\right), z\left(t-\Theta\right), u\left(t\right)\right), \\ y^2 + g\left(x\left(t\right), u\left(t\right)\right), t \in [t_0, t_0+\Theta], \\ \dot{x}(t) - f\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right)\right), \\ y^2 + g\left(x\left(t\right), u\left(t\right)\right), t \in [t_0+\Theta, t_1], \end{cases}$$

$$z = z(t) \in C^n[t_0-\Theta, t_0], x = x(t) \in W_{1,1}^n[t_0, t_1]$$

$$y = y(t) \in L_2^m[t_0, t_1], u = u(t) \in L_1^m[t_0, t_1].$$

The Lagrange function (see [3]) for the problem (13) – (18) has a form

$$L(w, \lambda_0, \lambda_1, ..., \lambda_n, y^*) = \sum_{i=0}^{n} \lambda_i f_i(w) + \langle y^*, F(w) \rangle$$

$$= \lambda_0[\int\limits_{t_0}^{t_0+\Theta} f^0\left(x\left(t\right), z\left(t-\Theta\right), u\left(t\right)\right) dt + \int\limits_{t_0+\Theta}^{t_1} f^0\left(x\left(t\right), x\left(t-\Theta\right), u\left(t\right)\right) dt]$$

$$+ \sum_{i=1}^{n} [\lambda_i(x^i(t) - x_1^i) + \int\limits_{t_0-\Theta}^{t_0} (z^i(t) - \hat{\varphi}^i(t) d\nu_i(t)$$

$$+\int_{t_0}^{t_0+\Theta}\psi(t)(\dot{x}(t)-f\left(x\left(t\right),z\left(t-\Theta\right),u\left(t\right)\right))dt$$

$$+\int_{t_0+\Theta}^{t_1}\psi(t)(\dot{x}(t)-f\left(x\left(t\right),x\left(t-\Theta\right),u\left(t\right)\right))dt+\int_{t_0}^{t_1}\mu(t)(y^2(t)+g(x(t),u(t))dt,$$

where $\nu_1(t),...,\nu_n(t)$- functions of the limited variation, $\psi(t)\in L^n_\infty[t_0,t_1],\mu(t)\in L^m_2[t_0,t_1]$. Using Theorem 1 from [4] we have: for any solution $\widehat{w}$ of the problem (13)-(18), there exist numbers $\lambda_i\geq 0, i=\overline{0,n}$ and on element y^* of the conjugate space Y^* such that the conditions:

a) $(\lambda_0,\lambda_1,...,\lambda_n,y^*)\neq(0,...,0)$,

b) $\lambda_i f_i(\widehat{w})=0, i=\overline{1,n}$,

c) $L(\widehat{w},\lambda_0,\lambda_1,...,\lambda_n,y^*)=\min_{w\in\ W}L(w,\lambda_0,\lambda_1,...,\lambda_n,y^*)$,

d) $\sum_{i=0}^n\lambda_i\frac{\partial f_i(\widehat{w})}{\partial w}+(F'(\widehat{w}))^*y^*=0$,

are fulfilled. From standard transformations (see [5]) of these conditions we receive conditions (6)-(12).

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