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ON ONE METHOD OF THE SOLUTION OF A NONLINEAR INTEGRO-DIFFERENTIAL EQUATION FOR A STRING

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Abstract. A boundary value problem is posed for an ordinary differential equation describing the static state of a string. The question as to the accuracy of one method of the solution of this problem is discussed.

Keywords and phrases: String static equation, bisection method, simple iteration method.

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Let us consider the boundary value problem

$$\varphi\left(\int_0^1 w'^2\,dx\right)w''=f,\quad 0<x<1,\tag{1}$$

$$w(0)=0,\quad w(1)=0,\tag{2}$$

characterizing the static state of a string. Here $w=w(x)$ is the displacement function we are seeking for, while of the given functions $f=f(x)$ and $\varphi=\varphi(z)$ the first one corresponds to the acting force and the second one is described by stress-strain relations. It is assumed that $\varphi=\varphi(z)$, $0\le z<\infty$, is a continuous or differentiable function that satisfies the condition

$$\varphi(z)\ge\alpha>0,\quad 0\le z<\infty.\tag{3}$$

When $\varphi(z)$ is a linear function, equation (1) is obtained from Kirchhoff's string oscillation equation by eliminating the time argument t [1]. The introduction of the function $\varphi(z)$ enables us not to restrict the consideration to Hooke's law in the stress-strain relation [2].

Let problem (1), (2) have a solution. To find it, we will use M. Chipot's approach [3]–[5]. The function we are seeking for is represented as the following product

$$w(x)=\lambda v(x),\tag{4}$$

where λ and $v(x)$ are respectively the parameter and the function to be found. The substitution of (4) into (1) gives

$$\lambda\varphi\left(\int_0^1(\lambda v')^2\,dx\right)v''=f.\tag{5}$$

Without loss of generality, equation (5) can be replaced by the system of equation

$$v'' = f,$$
$$\lambda\varphi\left(\int_0^1 (\lambda v')^2\,dx\right) = 1.$$

With (2) and (4) taken additionally into account, we come to a conclusion that for the function $v(x)$ we have the boundary value problem

$$v'' = f, \tag{6}$$
$$v(0) = 0, \quad v(1) = 0, \tag{7}$$

while the parameter λ is defined as a solution of the equation

$$\lambda\varphi(s\lambda^2) = 1, \tag{8}$$

where

$$s = \int_0^1 (v')^2\,dx. \tag{9}$$

The solution of problem (6), (7) has the form

$$v(x) = (x-1)\int_0^1 \zeta f(\zeta)\,d\zeta + x\int_x^1 (\zeta-1)f(\zeta)\,d\zeta. \tag{10}$$

As for the parameter s figuring in equation (8), from (9) and (10) it follows that its value is calculated by the formula

$$s = \int_0^1 \left[\int_0^x \zeta f(\zeta)\,d\zeta + \int_x^1 (\zeta-1)f(\zeta)\,d\zeta\right]^2 dx. \tag{11}$$

Let consider the question of the solution of equation (8). By (3) we conclude that its solution is a positive integer. Let us transform this equation. After squaring its both sides we get $\lambda^2\varphi^2(s\lambda^2) = 1$ and then multiply the obtained equality by s. As a result we get $s\lambda^2\varphi^2(s\lambda^2) = s$. If now we introduce the notation

$$\mu = s\lambda^2, \tag{12}$$

then the obtained equation takes the form

$$\mu\varphi^2(\mu) = s. \tag{13}$$

Let $\varphi(z)$ be a continuous function, $0 \le z < \infty$. We introduce the interval $I = \left[0, \dfrac{s}{\alpha^2}\right]$ into the consideration and define on it the function $g(\mu) = \mu\varphi^2(\mu) - s$. Let us replace (13) by the equation

$$g(\mu) = 0. \tag{14}$$

By virtue of (3) and (11), $g(0)g\Big(\dfrac{s}{\alpha^2}\Big) < 0$. Therefore equation (14) has a solution on I. To find it, we will use an approximate algorithm. Below we consider two possibilities: the bisection method and the method of simple iteration.

1. Bisection method. Denoting $a_0 = 0$, $b_0 = \dfrac{s}{\alpha^2}$ and applying this method [6], we obtain a sequence of intervals $[a_0, b_0]$, $[a_1, b_1], \dots$, $[a_n, b_n], \dots$, embedded into each other, for which $g(a_n)g(b_n) < 0$, $b_n - a_n = \dfrac{1}{2^n}(b_0 - a_0)$, $n = 0, 1, 2, \dots$. Hence we conclude that for the solution μ of equation (14) the inequality $0 \le \mu - a_n \le \dfrac{1}{2^n}\dfrac{s}{\alpha^2}$ is fulfilled. There exists a general limit $\lambda = \lim\limits_{n\to\infty}\left(\dfrac{a_n}{s}\right)^{\frac12} = \lim\limits_{n\to\infty}\left(\dfrac{b_n}{s}\right)^{\frac12}$ which is a solution of equation (8), and also we have

$$0 \le \lambda - \left(\frac{a_n}{s}\right)^{\frac12} \le \left(\frac{a_n}{s} + \frac{1}{2^n}\frac{1}{\alpha^2}\right)^{\frac12} - \left(\frac{a_n}{s}\right)^{\frac12}.$$

2. Simple iteration method. Assume that the function $\varphi(z)$ is differentiable on I. Rewrite equation (13) as $\mu = \dfrac{s}{\varphi^2(\mu)}$. Apply the iteration $\mu_{n+1} = \dfrac{s}{\varphi^2(\mu_n)}$, $n = 0, 1, \dots$ [6]. From condition (3) it follows that the function $\dfrac{s}{\varphi^2(\mu)}$ transform the interval I into itself. Let $2s\,\dfrac{|\varphi'(\mu)|}{\varphi^3(\mu)} \le q < 1$ be fulfilled for any μ from I. Then the iteration process converges for any initial approximation $\mu_0 \in I$, and $\lambda = \lim\limits_{n\to\infty}\left(\dfrac{\mu_n}{s}\right)^{\frac12}$ is a unique solution of equation (8), while for method error the following estimate

$$\left|\lambda - \left(\frac{\mu_n}{s}\right)^{\frac12}\right| \le \frac{q^n}{1-q}\left|\left(\frac{\mu_1}{s}\right)^{\frac12} - \left(\frac{\mu_0}{s}\right)^{\frac12}\right|$$

is true.

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