Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 22, 2008

ON THE COMPLEX REPRESENTATIONS FOR THE NONLINEAR AND NON-SHALLOW SHELLS

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Abstract. In the present paper under non-shallow shells will be meant three-dimensional shell-type elastic bodies satisfying the conditions $|hb_\alpha^\beta| \leq q < 1$ $(\alpha, \beta = 1, 2)$, in contrast to shallow shells, for which the assumption $hb_\alpha^\beta \cong 0$ is accepted, where h is the semi-thickness, b_α^β are mixed components of the curvature tensor of the shell's midsurface S.

Using the method I. Vekua [1] and the method of a small parameter [2] two-dimensional system of equations for the nonlinear and non-shallow shells is obtained [3]. For approximations $N = 0, 1, 2, 3$ the complex representations of the general solutions and boundary conditions are obtained.

Keywords and phrases: Non-shallow shells, metric tensor and tensor of curvature, midsurface of the shell.

AMS subject classification: 74K25; 74B20.

1. The equilibrium equation of the continuous medium and stress-strain relations have the form:

$$\nabla_i \boldsymbol{\sigma}^i + \boldsymbol{\Phi} = \mathbf{0}, \quad \boldsymbol{\sigma}^{\mathbf{i}} = \tau^{\mathbf{ij}}(\mathbf{R_j} + \partial_{\mathbf{j}}\mathbf{u}) \quad \left(\partial \mathbf{j} = \frac{\partial}{\partial \mathbf{x^j}}\right),$$

$$\tau^{ij} = \lambda \Theta g^{ij} + 2\mu e^{ij} \quad (\mathbf{R_j} = \partial_{\mathbf{j}}\mathbf{R}, \quad \mathbf{g^{ij}} = \mathbf{R^i R^j}),$$

$$(i, j = 1, 2, 3),$$

where ∇_i are covariant derivatives with respect to the space coordinates x^i, $\boldsymbol{\sigma}^i$ are covariant constituents of the stress tensor, $\boldsymbol{\Phi}$ is vector of volume forces, τ^{ij} and e^{ij} are contravariant components of the stress and strain tensors, respectively,

$$e^{ij} = \frac{1}{2}(\mathbf{R^i}\partial^{\mathbf{j}}\mathbf{u}^{+}\mathbf{R^j}\partial^{\mathbf{i}}\mathbf{u} + \partial^{\mathbf{i}}\mathbf{u}\partial^{\mathbf{j}}\mathbf{u}), \quad \boldsymbol{\Theta} = \mathbf{R^i}\partial_{\mathbf{i}}\mathbf{u} + \frac{\mathbf{1}}{\mathbf{2}}\partial^{\mathbf{i}}\mathbf{u}\partial_{\mathbf{i}}\mathbf{u} \quad (\partial^{\mathbf{j}} = \mathbf{g^{ij}}\partial_{\mathbf{j}}).$$

The two-dimensional finite system of equilibrium equations with respect to component of displacement vector in the isometric coordinates has the complex form [3]:

$$4\mu\partial_{\bar z}\left(\Lambda^{-1}\partial_z \overset{(m)}{u}_+\right) + 2(\lambda+\mu)\partial_{\bar z}\overset{(m)}{\theta} + 2\lambda\partial_{\bar Z}\overset{(m)}{u'_3}$$
$$-(2m+1)\left[2\partial_{\bar z}\left(\overset{(m-1)}{u_3} + \overset{(m-3)}{u_3} + \cdots\right) + \overset{(m-1)}{u'_+} + \overset{(m-3)}{u'_+} + \cdots\right] = \overset{(m)}{F}_+ ,$$

$$\mu\Big(\nabla^2\overset{(m)}{u_3}+\overset{(m)}{\theta'}\Big)-(2m+1)\Big[\lambda\Big(\overset{(m-1)}{\theta}+\overset{(m-3)}{\theta}+\cdots\Big) \tag{1}$$
$$+(\lambda+2\mu)\Big(\overset{(m-1)}{u'_3}+\overset{(m-3)}{u'_3}+\cdots\Big)\Big]=\overset{(m)}{F_3}\,,$$
$$\Big(m=0,1,\ldots,N;\ \ \overset{(k)}{u_i}=0,\ k>N,\ \ \overset{(m)}{u_+}=\overset{(m)}{u_1}+i\overset{(m)}{u_2}\Big),$$

where

$$\overset{(m)}{u_i}=\frac{2m+1}{2h}\int_{-h}^{h}u_iP_m\Big(\frac{x_3}{h}\Big)dx_3,\ \ \overset{(m)}{u'_i}=\frac{2m+1}{h}\Big(\overset{(m+1)}{u_i}+\overset{(m+3)}{u_i}+\cdots\Big),$$
$$\overset{(m)}{\theta}=2\,\mathrm{Re}\,\Big(\Lambda^{-1}\partial_z\overset{(m)}{u_+}\Big),\ \ z=x^1+ix^2,\ \ 2\partial_z=\partial_1-i\partial_2,$$
$$\nabla^2=4\Lambda^{-1}\partial^2_{z\overline{z}}\,,\ \ ds^2=\Lambda(z,\overline{z})dz\,d\overline{z},$$

P_m is the Legendre polynomial of degree m, the right-hand sides $\overset{(m)}{F_+}=\overset{(m)}{F_1}+i\overset{(m)}{F_2}$ and $\overset{(m)}{F_3}$ are expressed by means of previous approximations.

2. Consider now the cases: $N=0,1,2,3$.

Case $N=0$. From (3) we get:

$$4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z\overset{(0)}{u_+}\Big)+2(\lambda+\mu)\partial_{\overline{z}}\overset{(0)}{\theta}=0,\ \ \Big(\overset{(m)}{F_+}=\overset{(m)}{F_3}=0,\ m=0,1,2,3\Big),$$
$$\mu\nabla^2\overset{(0)}{u_3}=0.$$

The complex representation of general solutions has the form:

$$\overset{(0)}{u_+}=-\frac{\lambda+3\mu}{\lambda+\mu}\,\frac{1}{\pi}\iint_S\frac{\varphi'(\zeta)dS}{\overline{\zeta}-\overline{z}}+\frac{1}{\pi}\iint_S\frac{\overline{\varphi'(\zeta)}dS}{\overline{\zeta}-\overline{z}}-\overline{\psi(z)},$$
$$\overset{(0)}{u_3}=f(z)+\overline{f(z)},\ \ \Big(dS=\Lambda(\zeta,\overline{\zeta})d\zeta\,d\overline{\zeta},\ \ \zeta=\xi+i\eta\Big),$$

where $f(z)$, $\varphi(z)$ and $\psi(z)$ are holomorphic functions of z. We note that for plane (i.e. $\Lambda=1$) the expression of $\overset{(0)}{u_+}$ coincides with the well-known representation of Kolosov-Muskhelishvili.

Case $N=1$. With respect to the components $(\overset{(0)}{u_+},\overset{(1)}{u_3})$ and $(\overset{(1)}{u_1},\overset{(0)}{u_3})$ we have two systems of equations:

$$\begin{aligned}&4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z\overset{(0)}{u_+}\Big)+2(\lambda+\mu)\partial_{\overline{z}}\overset{(0)}{\theta}+2\lambda\partial_{\overline{z}}\overset{(1)}{u_3}=0,\\&\mu\nabla^2\overset{(1)}{u_3}+3\Big[\lambda\overset{(0)}{\theta}+(\lambda+2\mu)\overset{(1)}{\theta}\Big]=0\end{aligned} \tag{2}$$

and

$$\begin{aligned}&4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z\overset{(1)}{u_+}\Big)+2(\lambda+\mu)\partial_{\overline{z}}\overset{(1)}{\theta}-3\mu\Big(2\partial_{\overline{z}}\overset{(0)}{u_3}+\overset{(1)}{u_+}\Big)=0,\\&\nabla^2\overset{(0)}{u_3}+\overset{(1)}{\theta}=0.\end{aligned} \tag{3}$$

The complex representation of general solutions has the form:

$$\overset{(0)}{u}_{+}=-\frac{5\lambda+6\mu}{3\lambda+2\mu}\,\frac{1}{\pi}\iint_S\frac{\varphi'(\zeta)dS}{\overline{\zeta}-\overline{z}}+\frac{1}{\pi}\iint_S\frac{\overline{\varphi'(\zeta)}dS}{\overline{\zeta}-\overline{z}}-\overline{\psi(z)}-\frac{\lambda}{6(\lambda+\mu)}\,\frac{\partial v}{\partial\overline{z}}\,,$$
$$\overset{(0)}{u}_{3}=v-\frac{2\lambda}{\lambda+2\mu}(\varphi'+\overline{\varphi'}),\quad\Big(\nabla^2v=\frac{12(\lambda+\mu)}{\lambda+2\mu}\,v\Big),$$

and

$$\overset{(1)}{u}_{+}=-\frac{1}{\pi}\iint_S\frac{\Phi'(\zeta)+\overline{\Phi'(\zeta)}}{\overline{\zeta}-\overline{z}}dS+\frac{4(\lambda+\mu)}{3\mu}\overline{\Phi''(z)}-2\overline{\Psi'(z)}+i\frac{\partial\chi}{\partial\overline{z}}\,,$$
$$\overset{(0)}{u}_{3}=\Psi(z)+\overline{\Psi(z)}-\iint_S\Big(\Phi'(\zeta)+\overline{\Phi'(\zeta)}\Big)\ln|\zeta-z|dS\ \ (\nabla^2\chi=3\chi),$$

where $\Phi(z)$ and $\Psi(z)$ are holomorphic functions of z.

Note that the systems (2) and (3) coincides with I. Vekua's refined systems of equations for the stretch-strain and bending of plate, respectively.

Case $N=2$. In this case with respect to the components $(\overset{(0)}{u}_1,\overset{(2)}{u}_+,\overset{(1)}{u}_3)$ and $(\overset{(1)}{u}_+,\overset{(0)}{u}_3,\overset{(2)}{u}_3)$ we have two system of equations:

$$\begin{aligned}
&4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z\overset{(0)}{u}_{+}\Big)+2(\lambda+\mu)\partial_{\overline{z}}\overset{(0)}{\theta}+2\lambda\partial_{\overline{z}}\overset{(1)}{u}_3=0,\\
&4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z\overset{(2)}{u}_{+}\Big)+2(\lambda+\mu)\partial_{\overline{z}}\overset{(2)}{\theta}-5\mu\Big(2\partial_{\overline{z}}\overset{(1)}{u}_3+3\overset{(2)}{u}_{+}\Big)=0,\\
&\mu\Big(\nabla^2\overset{(1)}{u}_3+3\overset{(2)}{\theta}\Big)-3\Big[\lambda\overset{(0)}{\theta}+(\lambda+2\mu)\overset{(1)}{u}_3\Big]=0
\end{aligned}\tag{4}$$

and

$$\begin{aligned}
&4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z\overset{(1)}{u}\Big)+2(\lambda+\mu)\partial_{\overline{z}}\overset{(1)}{\theta}-3\mu\Big(2\partial_{\overline{z}}\overset{(0)}{u}_3+\overset{(1)}{u}_{+}\Big)=0,\\
&\nabla^2\overset{(0)}{u}_3+\overset{(1)}{\theta}=0,\\
&\mu\nabla^2\overset{(2)}{u}_3-5\Big[\lambda\overset{(1)}{\theta}+3(\lambda+2\mu)\overset{(2)}{u}_3\Big]=0.
\end{aligned}\tag{5}$$

In this case the complex representations of the general solutions of the equations (4) and (5) take the forms

$$\overset{(0)}{u}_{+}=-\frac{5\lambda+6\mu}{3\lambda+2\mu}\,\frac{1}{\pi}\iint_S\frac{\varphi'(\zeta)dS}{\overline{\zeta}-z}+\frac{1}{\pi}\iint_S\frac{\overline{\varphi'(\zeta)}dS}{\overline{\zeta}-\overline{z}}-\overline{\psi(z)}-\frac{2\lambda}{\lambda+2\mu}\sum_{k=1}^{2}\frac{1}{\alpha_k}\,\frac{\partial v_k}{\partial\overline{z}}\,,$$
$$\overset{(2)}{u}_{+}=\frac{2}{3}\Big(i\frac{\partial\omega}{\partial\overline{z}}+\frac{2\lambda}{3\lambda+2\mu}\,\overline{\varphi''(z)}+\sum_{k=1}^{2}\frac{\alpha_{3-k}}{\alpha_k}\,\frac{\partial v_k}{\partial\overline{z}}\Big),$$
$$\overset{(1)}{u}_{3}=v_1+v_2-\frac{2\lambda}{\lambda+2\mu}\big(\varphi'(z)+\overline{\varphi'(z)}\big),$$

where

$$\nabla^2v_k=\alpha_kv_k,\ \ \alpha_k^2-\frac{12(\lambda+\mu)}{\lambda+2\mu}\,\alpha_k+\frac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2}=0\ \ (k=1,2);\ \ \nabla^2\omega=15\omega,$$

and

$$\overset{(1)}{u}_+ = -\frac{1}{\pi}\iint_S \frac{\Phi'(\zeta)+\overline{\Phi'(\zeta)}}{\overline{\zeta}-\overline{z}}dS + \frac{16(\lambda+\mu)}{3(\lambda+2\mu)}\,\overline{\Phi''(z)} - 2\overline{\Psi'(z)} + i\frac{\partial\chi}{\partial\overline{z}} - \frac{\lambda}{10(\lambda+\mu)}\,\frac{\partial w}{\partial \overline{z}}\,,$$

$$\overset{(0)}{u}_3 = \Psi(z)+\overline{\Psi(z)} - \frac{1}{\pi}\iint_S\Big(\Phi'(\zeta)+\overline{\Phi'(\zeta)}\Big)\ln|\zeta-z|dS + \frac{\lambda}{20(\lambda+\mu)}\,w,$$

$$\overset{(2)}{u}_3 = w - \frac{2\lambda}{3\lambda+2\mu}\Big(\Phi'(z)+\overline{\Phi'(z)}\Big),$$

where $\nabla^2 w = \frac{60(\lambda+\mu)}{\lambda+2\mu}\,w$, $\nabla^2\chi = 3\chi$.

Case $N=3$. For this case we have

$$\overset{(0)}{u}_+ = -\frac{5\lambda+6\mu}{3\lambda+2\mu}\,\frac{1}{\pi}\iint_S\frac{\varphi' dS}{\overline{\zeta}-\overline{z}} + \frac{1}{\pi}\iint_S\frac{\overline{\varphi'}dS}{\overline{\zeta}-\overline{z}} - \overline{\psi} - \frac{2\lambda}{\lambda+2\mu}\sum_{k=1}^{3}\frac{1+\overset{(1)}{A}_k}{\alpha_k}\,\frac{\partial v_k}{\partial\overline{z}}\,,$$

$$\overset{(2)}{u}_+ = \frac{2}{3}\bigg(i\frac{\partial\omega}{\partial\overline{z}} + \frac{2\lambda}{3\lambda+2\mu}\,\overline{\varphi''} + \sum_{k=1}^{3}\overset{(2)}{A}_k\frac{\partial v_k}{\partial\overline{z}}\bigg),$$

$$\overset{(1)}{u}_3 = \sum_{k=1}^{3}\overset{(1)}{A}_k v_k - \frac{2\lambda}{3\lambda+2\mu}(\varphi'+\overline{\varphi'}),$$

$$\overset{(3)}{u}_3 = \sum_{k=1}^{3} v_k, \quad \Big(\nabla^2 v_k = \alpha_k v_k, \quad \nabla^2\omega = 15\omega\Big),$$

where

$$\alpha_k^3 - \frac{180(\lambda+\mu)}{\lambda+2\mu}\,\alpha_k^2 + \frac{120(\lambda+\mu)(7\lambda+15\mu)}{(\lambda+2\mu)^2}\,\alpha_k + \frac{7\cdot 900(\lambda+\mu)}{\lambda+2\mu} = 0,$$

$$\overset{(1)}{A}_k = \bigg[\frac{3(9\lambda+4\mu)}{\lambda+2\mu}\,\alpha_k - \frac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\bigg]\bigg[\alpha_k^2 - \frac{12(\lambda+\mu)}{\lambda+2\mu}\,\alpha_k + \frac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\bigg]^{-1},$$

$$\overset{(2)}{A}_k = -10\bigg[\frac{\lambda}{\lambda+2\mu}\,\alpha_k - \frac{12(\lambda+\mu)}{\lambda+2\mu}\bigg]\bigg[\alpha_k^2 - \frac{12(\lambda+\mu)}{\lambda+2\mu}\,\alpha_k + \frac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\bigg]^{-1},$$

and

$$\overset{(1)}{u}_+ = -\frac{1}{\pi}\iint_S\frac{\Phi'+\overline{\Phi'}}{\overline{\zeta}-\overline{z}}\,dS + \frac{4}{15}\,\frac{23\lambda+24\mu}{\lambda+2\mu}\overline{\Phi''} - 2\overline{\Psi} + \sum_{k=1}^{2}\bigg(i\frac{\varkappa_k-3}{3}\,\frac{\partial\chi_k}{\partial\overline{z}} - \frac{6\lambda}{\lambda+2\mu}\,\frac{\partial w_k}{\partial\overline{z}}\bigg),$$

$$\overset{(3)}{u}_+ = \sum_{k=1}^{2}\bigg(i\frac{\partial\chi_k}{\partial\overline{z}} + \frac{2}{5}\,\frac{\gamma_{3-k}}{\gamma_k}\,\frac{\partial w_k}{\partial\overline{z}}\bigg) - \frac{4}{15}\,\frac{3\lambda+2\mu}{\lambda+2\mu}\,\overline{\Phi''}(z),$$

$$(\nabla^2 w_k = \gamma_k w_k, \quad \nabla^2\chi_k = \varkappa_k\chi_k,),$$

$$\overset{(0)}{u}_3 = \sum_{k=1}^{2} \left(\frac{3\lambda}{\lambda + 2\mu} - \frac{\gamma_{3-k}}{5} \right) \frac{1}{\gamma_k} w_k - \iint_S (\Phi' + \overline{\Phi'}) \ln |\zeta - z| dS + \Psi + \overline{\Psi},$$

$$\overset{(2)}{u}_3 = \sum_{k=1}^{2} w_k - \frac{2\lambda}{3(\lambda + 2\mu)} (\Phi' + \overline{\Phi'})$$

$$\left(\gamma_k^2 - \frac{60(\lambda + \mu)}{3(\lambda + 2\mu)} \gamma_k + 60 \frac{35\mu(\lambda + \mu)}{(\lambda + 2\mu)^2} = 0, \quad \varkappa_k^2 - 45\varkappa_k + 105 = 0 \right).$$

3. For the first boundary condition (in stress) we have

$$(\lambda + \mu) \overset{(m)}{\theta} - 2\mu\Lambda \frac{\partial \overset{(m)}{u}_+}{\partial \overline{z}} \left(\frac{d\overline{z}}{ds} \right)^2 = \overset{(m)}{a}_1 + i \overset{(m)}{b}_1, \quad \operatorname{Im} \left(\frac{\partial \overset{(m)}{u}_3}{\partial \overline{z}} \frac{d\overline{z}}{ds} \right) = \overset{(m)}{c}_1 \quad (\text{on } \partial S).$$

The second boundary condition (in displacements) for any m takes the form

$$\overset{(m)}{u}_+ \frac{d\overline{z}}{ds} = \overset{(m)}{a}_2 + i \overset{(m)}{b}_2, \quad \overset{(m)}{u}_3 = \overset{(m)}{c}_2 \quad (\text{on } \partial S) \quad (m = 0, 1, 2, 3).$$

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Received 30.07.2008; revised 12.12.2008; accepted 24.12.2008.