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ABOUT THE USE OF AN APPROACH ALTERNATIVE TO THE ASYMPTOTIC METHOD FOR LINEAR NONHOMOGENEOUS BOUNDARY PROBLEMS

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Abstract. Problems of approximate solution of some linear nonhomogeneous operator equation is studied with an approach alternative to asymptotic method. Our alternative method is based on representation of unknown vector over the small parameter with orthogonal series instead of asymptotic one. In such a case system of three-point operator equations of special structure is received. for system solving a certain regular method is used. On the basis of the suggested method the programming production is created and realized by means of computer.

Keywords and phrases: Nonhomogeneous operator equation, asymptotic series, orthogonal series, asymptotic method, alternative method.

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Introduction. Perturbation theory is a method to study and solve topical problems of science and technology. It comprises mathematical methods that are used to find an approximate solution to a problem which cannot be solved exactly; among them Poincare-Lyapunov's method, known as a small parameter method, is widely applied as one of the most powerful methods of research and calculation, but its convergence is asymptotic. While using the asymptotic method we obtain a double-point recurrent system of equations. In the present paper, problems of approximate solution of some boundary value problems are studied by means of a numerical-experimental method based on an approach, alternative to the asymptotic method. The method is developed by prof. T. Vashakmadze [1, p. 124-127]. It is based on presentation of the required vector with an orthogonal series instead of asymptotic one. In that case we obtain three-point recurrent system of operator equations of the special structure. A solution of the system is obtained inverting a relatively simple operator N-times and acting with the operator describing a perturbation degree over the known magnitudes N-times. The degree N of the polynomial defines exactness of the method. An asymptotic method converges close to zero; in addition, a radius of convergence is to be determined, while an alternative method converges in the fixed interval.

Algorithms of approximate solution of a linear nonhomogeneous operator equation are considered using both asymptotic and its alternative methods in [2]. There were shown general formulation of both methods and essential differences between them. Their comparative analysis was made. In the present paper the above-mentioned methods are approved for problems of finding an approximate solution of a double-point boundary problem with variable coefficients and that of an integro-differential equation.

1. Approximate solving of double-point boundary value problem by an approach alternative to the asymptotic method. Let us consider a double-point boundary value problem with variable coefficients

$$\begin{cases} u''(x) - q(x)u(x) = -f(x), & x \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $q(x) \geq 0, q(x), f(x) \in L_2[0,1]$.

Problem (1.1) has an unique solution in the class $W^2_{2,0}(0,1)$. Using Green function it may be presented in the following form:

$$u(x) = \int_0^1 G(x,\xi,q(\cdot))f(\xi)d\xi. \tag{1.2}$$

Notation $G(x,\xi,q(\cdot))$ shows that Green function $G(x,\xi)$ of given boundary value problem depends on the function $q(x)$ (on the variable coefficient), i.e. the Green function may be considered as nonlinear operator in respect with q. It is shown in [3] that at certain conditions $G(x,\xi,q(\cdot))$ is analytic in the sense of Gateaux at the point $q = 0$, the corresponding operator series is uniformly convergent, i.e. we can approach to the Green function for any $q(x)$ by the well known Green functions constructed for $q(\cdot) = 0$.

$$G(x,\xi,0) = G(x,\xi) = \begin{cases} x(1-\xi), & 0 \leq x \leq \xi, \\ \xi(1-x), & \xi \leq x \leq 1. \end{cases} \tag{1.3}$$

The above-mentioned boundary value problem was solved with an operator-interpo lation approach [4]. In particular, new algorithms have been considered for an approximate solution of a double-point boundary value problem with variable coefficients. Approximation to the Green function of the given boundary value problem was made using operator interpolation polynomials of the Newton type (here linear combinations of the Heaviside function are used). Experimental convergence is revealed in respect to the degree of interpolation polynomials and a node number while approximate integration.

In the present paper, boundary problem (1.1) is solved by asymptotic method (A) and by an approach, alternative to the asymptotic one (B). In our case the basic operator is $Lu = u''$ and operator describing the perturbation degree is $Mu = q_1 u,\ q_1 = q/\varepsilon$.

Approximate solution formulae for approximation of the inverted operator are constructed:

$$\nu_0(x) = \int_0^1 G(x,\xi,0)f_0(\xi)d\xi; \quad \nu_i(x) = -\int_0^1 G(x,\xi,0)q_1(\xi)\nu_{i-1}(\xi)d\xi, \quad i = 1,2,3.$$

$$w_0(x) = \nu_0(x) + \frac{1}{3}\nu_2(x),\ w_2(x) = \frac{2}{3}\nu_2(x), \quad w_1(x) = \nu_1(x) + \frac{3}{5}\nu_3(x), \quad w_3(x) = \frac{2}{5}\nu_3(x).$$

2. Approximate solving of some linear nonhomogeneous integro-differential equations by an approach alternative to the asymptotic method. Let us consider the following integro-differential equation:

$$\begin{cases} u''(x) - \varepsilon \int\limits_0^1 K(x,t)u(t)dt = -f(x), \\ u(0) = u(1) = 0, \end{cases} \tag{2.1}$$

$$f(x) \in L_2[0,1], |K(x,t)| \leq M \leq +\infty, \quad K(x,t) \in C([0,1] \times [0,1]).$$

Problem (2.1) has a unique solution in the class $W^2_{2,0}(0,1)$. It can be found by the successive approximation method. The approach elaborated in the §1 enables us to find and represent an approximate solution of equation (2.1) applying the Green formulae. Let us solve the above-mentioned problem by both asymptotic and alternative methods. In our case the basic operator is $Lu = u''$ and operator, describing the perturbation degree, is $Mu = \int\limits_0^1 K(x,t)u(t)dt$. Analogously §1, here, in the case of asymptotic method the approximation formulae of the inverted operator are constructed:

$$\begin{aligned} \nu_0(x) &= \int\limits_0^1 G(x,\xi,0)f_0(\xi)d\xi, \\ \nu_i(x) &= \int\limits_0^1 G(x,\xi,0)\int\limits_0^1 K(\xi,t)\nu_{i-1}(t)dtd\xi \quad i = 1,2,3,\cdots. \end{aligned} \tag{2.2}$$

The algorithm proposed in [2] enables us to find approximate solutions of problems (1.1) and (2.1) for $N = 2$ and $N = 4$ as by the asymptotic method so by the alternative approach. Of course, the formulae connecting the (A) and (B) methods are in effect. To calculate multiple integrals with predetermined exactness we use Simpson's quadrature formulae.

For approximate solving boundary value problem the complex of programs in algorithm language *Turbo Pascal* is composed and many numerical experiments are carried out. The results obtained are good enough. In practical tasks it is often sufficient if we take interpolation polynomial or operator series of the order not more than 2 (see [3], [4]). Also, in the case of applying asymptotic and alternative methods $N \leq 4$ is quite enough.

For clearness let us consider a test problem when $K(x,t) = e^{xt}$, an exact solution $u(x) = x(1-x)$; the right hand side $f(x) = 2 + e^x\left[\frac{1}{x^2} - \frac{2}{x^3}\right] + \frac{1}{x^2} + \frac{2}{x^3}$; a small parameter $\varepsilon = 0.001$; a node $x = 0.5$; n - number of the interval divisions. Numerical results obtained are given in the Table below:

	n=10	n=20	n=40	n=80
$u_1[\nu]$	0.24184802	0.23731695	0.23629459	0.23618693
$u_1[w]$	0.24184802	0.23731695	0.23629459	0.23618693
$u_3[\nu]$	0.24640284	0.24213295	0.24133153	0.24127600
$u_3[w]$	0.24640284	0.24213295	0.24133153	0.24127600
u_{exact}	0.25	0.25	0.25	0.25

where

$$u_1[w] = w_0 + \varepsilon w_1$$
$$u_3[w] = w_0 + \varepsilon w_1 + P_2(\varepsilon) w_2 + P_3(\varepsilon) w_3$$
$$u_1[\nu] = \nu_0 + \varepsilon \nu_1$$
$$u_3[\nu] = \nu_0 + \varepsilon \nu_1 + \varepsilon^2 \nu_2 + \varepsilon^3 \nu_3$$

u_{exact} – the exact value of function at the given point.

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