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APPLICATION OF THE ORTHOGONAL INVARIANTS OF THREE-DIMENSIONAL OPERATORS IN SOME HYDRODYNAMICAL PROBLEMS

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Abstract. On the base of complete set of orthogonal invariants of operators which is built in the recently performed series of articles of one of the authors (I.L.) the system of nonlinear differential equations (NDE) for orthogonal invariants of ideal gas (liquid) hydrodynamical velocity's Jacobi matrix is obtained when the flow is barochronic. It is shown that only two regimes of barochronic flow are possible - a potential and/or a solenoidal one. Exact solutions of the NDE system are obtained; there are found polynomial relations between the Jacobi matrix's invariants and it is proved that these relations are integrals of motion. Using the obtained results the 3-dimensional hydrodynamical Euler equations are solved and hydrodynamical velocity's and substance density's time- and space-dependence are found. It is shown that the hydrodynamical velocity of potential barochronic flow depends on radius-vector (for arbitrary chosen origin of coordinates frame) which satisfies the non-relativistic Hubble low. This result seems interesting taking into consideration that barochronic flow naturally describes long-scale evolution of the Universe. The sufficient and necessary conditions are find for the solution of hydrodynamical Euler equations of solenoidal barochronic stream having form of the primitive wave or of the double wave. It is shown that solenoidal barochronic flow is isobaric.

Keywords and phrases: Operators invariants, Barochronic flow, Hubble law.

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In the series of articles [1-4] there has been constructed a complete set of polynomial invariants - a polynomial basis of invariants of linear operators and of their matrices for orthogonal transformations in n-dimensional Euclidean space $\mathbb{E}^n$. The classification problem of the operators by their orthogonal invariants has been solved (in these studies the corresponding theorems for the unitary transformations group in n-dimensional unitary vector space $\mathbb{U}^n$ have been proved too).

The method developed can be successfully used in various physics problems. The results obtained in this way usually are more correct and detailed then obtained by other authors with methods differ from our ones. Our method allows finding some new solutions [5] that were not found by methods used previously [6]-[9].

In present study, the orthogonal invariants of three-dimensional matrix are used to solve three-dimensional nonlinear equation in partial derivatives, describing some hydro- and aero-dynamical problems. The similar method has been used in [6-9] where the differential equations (DE) system was obtained for algebraic invariants of Jacoby matrix of hydrodynamical velocity field. In [6-9] authors investigate types of symmetries of general solutions of the DE system for some hydrodynamical models. But the set of algebraic invariants used in [6-9] is not complete. In [5] we have shown that the application of complete polynomial basis of orthogonal invariants gets all (smooth) solutions of corresponding physics problems in covariant form. In common case the number of arbitrary parameters in obtained solutions can't be reduced.

1. Below we use the following main results of studies [1-4]:

I. For arbitrary linear operator $\mathbf{T}$ in real three-dimensional Euclidean space $\mathbb{E}^3$ there exists orthonormal canonical basis (CB) determined uniquely up to the simultaneous reflection of all coordinate axis, such that the corresponding matrix $T = [T_{ik}]$ (generally non-symmetric) of this operator gets the form

$$T = \begin{bmatrix} s_1 & \omega_3 & -\omega_2 \\ -\omega_3 & s_2 & \omega_1 \\ \omega_2 & -\omega_1 & s_3 \end{bmatrix}$$

$$(s_1 \leq s_2 \leq s_3, T_{12} \equiv \omega_3 \geq 0, T_{13} \equiv -\omega_2 \geq 0).$$

If $s_1 = s_2 < s_3$ (or if $s_1 < s_2 = s_3$) then applying the corresponding rotation in the coordinate plane x_1Ox_2 (or in the plane x_2Ox_3) that does not change the significance $T_{12} \equiv \omega_3$ ($T_{23} \equiv \omega_1$) the entry T_{13} always may become 0; if $s_1 = s_2 = s_3$ then the entries T_{13} and T_{23} may become 0.

II. The complete set of algebraic (polynomial) orthogonal invariants of given operator T has been built, which is reciprocally (biectally) connected with the entries of the matrix T of this operator in the CB. In three-dimensional real Euclidean space this set in common case consists from the following six polynomial invariants ($\mathrm{tr}M = \Sigma M_{kk}$) :

$$\mathrm{tr}S^\lambda, \ \ (\lambda = 1, 2, 3) \ \ \mathrm{tr}A^2, \ \ \mathrm{tr}(SA^2), \ \ \mathrm{tr}(SAS^2A^2), \tag{1}$$

where

$$S_{lk} = S_{kl} = \frac{1}{2}\,(T_{lk} + T_{kl}), \quad A_{lk} = -A_{kl} = \frac{1}{2}\,(T_{lk} - T_{kl}).$$

It is easy to show (see [1] and [5]), that in CB one has

$$\mathrm{tr}S^\lambda = s_1^\lambda + s_2^\lambda + s_3^\lambda, \qquad \lambda = 1, 2, 3;$$

$$\mathrm{tr}(S^\mu A^2) = s_1^\mu \omega_1^2 + s_2^\mu \omega_2^2 + s_3^\mu \omega_3^2 - \vec{\omega}^2 \mathrm{tr}S^\mu, \qquad \mu = 1, 2;$$

$$\mathrm{tr}(SAS^2A^2) = \omega_1\omega_2\omega_3 \det[s_k^{l-1}]_{k,l=1,2,3} = \omega_1\omega_2\omega_3(s_2 - s_1)(s_3 - s_1)(s_3 - s_2).$$

Hence two operators are orthogonally similar iff they have coincident set of invariants (1).

2. The method developed we use [5] to the problem of barochronic flow of ideal gas.

Definition. The flow of continued medium is called *barochronic* if its pressure P and the density ρ depend on time only and are the same in every point of Euclidean space $\mathbb{E}^3$ [8].

It is clear that three dimensional barochronic motions physically can be fulfilled only in infinite uniform space. So the investigation of such regime is rather interesting from the cosmological point of view because it gives us the possibility to separate effects caused by gravitation (curveting of the space) from the purely kinematical effects in plane infinite uniform (homogeneous) space.

It is also evident that while the gradient of pressure and density are zero the motion can happen only by inertia and the non trivial dependence of hydrodynamical velocity on coordinates and time is caused by the initial field of velocity that is considered to be continuous and sufficiently smooth.

For barochronic flow of ideal gas the Euler equation and the equation of continuity [10] take the form of non-linear differential equations [6], [7]

$$\partial_t u_k + u_j \partial_j u_k = 0, \quad \partial_t \rho + \rho \partial_j u_j = 0, \quad (k = 1, 2, 3)$$

for the components of the vector of hydrodynamical velocity $\vec{u} = (u_1, u_2, u_3)$ and density

$$\rho = \rho(t);\ \partial_t \equiv \frac{\partial}{\partial t},\ \partial_j \equiv \frac{\partial}{\partial x_j}\ (j = 1, 2, 3).$$

As usually, the summarization is meant by repeated indices.

Splitting the Jacoby matrix $J(t, \vec{x}) = [\partial_l u_k]_{k,\, l=1,2,3}$ on the symmetry and skew-symmetry parts

$$J(t, \vec{x}) = S + A,\ \ S = \frac{1}{2}[\partial_l u_k + \partial_k u_l]_{k,\, l=1,2,3},\ \ A = \frac{1}{2}[\partial_l u_k - \partial_k u_l]_{k,\, l=1,2,3},$$

we have shown in [5] that the following statements are fulfilled:

Proposition 1. The dependence of Jacoby matrix on time is determined by the equation

$$J(t, \vec{x}) = J(0, \vec{x})(E_3 + tJ(0, \vec{x}))^{-1}, \tag{2}$$

where E_3 denotes unique 3×3 matrix, and the matrix $(E_3 + tJ(0, \vec{x}))$ is not singular.

Theorem 1. *All algebraic invariants of Jacoby matrix $J(t, \vec{x}) = [\partial_k u_l]$ are time dependent only and the next correlations are valid:*

$$\left(\frac{\partial}{\partial t}\right)^m \mathrm{tr}[J(t, \vec{x})]^l = (-1)^m l(l+1)\dots(l+m-1)\mathrm{tr}[J(t, \vec{x})]^{l+m},$$

$$\frac{\partial}{\partial t}\mathrm{tr}(S^\lambda A^\zeta S^\mu A^\eta) = -(\lambda+\eta)\mathrm{tr}(S^{\lambda+1} A^\zeta S^\mu A^\eta)$$
$$-(\mu+\zeta)\mathrm{tr}(S^\lambda A^\zeta S^{\mu+1} A^\eta) - \lambda \mathrm{tr}(S^{\lambda-1} A^{\zeta+2} S^\mu A^\eta) - \mu \mathrm{tr}(S^\lambda A^\zeta S^{\mu-1} A^{\eta+2})$$
$$-\zeta \mathrm{tr}(S^\lambda A^{\zeta-1} SAS^\mu A^\eta) - \eta \mathrm{tr}(S^\lambda A^\zeta S^\mu A^{\eta-1} SA).$$

$$(l, m, \lambda, \mu, \zeta, \eta \in \mathbb{N})$$

Theorem 2. *The elements of polynomial basis of invariants (1) satisfy the closed system of ordinary differential equations* ($f' \equiv \partial_t f,\ S_1 \equiv \mathrm{tr}S = \mathrm{tr}(S+A) = \mathrm{div}\,\vec{u}(t,\vec{x})$, $\gamma \equiv -\mathrm{tr}A^2 = (\mathrm{rot}\,\vec{u})^2/2$:

$$
\begin{aligned}
&(\mathrm{tr}S)' = -\mathrm{tr}S^2 - \mathrm{tr}A^2;\\
&(\mathrm{tr}A^2)' = -4\mathrm{tr}(SA^2);\\
&(\mathrm{tr}S^2)' = -2\mathrm{tr}S^3 - 2\mathrm{tr}(SA^2);\\
&(\mathrm{tr}S^3)' = -3\mathrm{tr}S^4 - 3\mathrm{tr}(S^2A^2);\ (\mathrm{tr}S^4 = S_1^4/6 + 4/3S_1\mathrm{tr}S^3 - (S_1^2 - \mathrm{tr}S^2/2)\mathrm{tr}S^2)\\
&\mathrm{tr}(SA^2)' = \mathrm{tr}(S^2A^2) - 4S_1\mathrm{tr}(SA^2) + (S_1^2 - \mathrm{tr}S^2)\mathrm{tr}A^2 - (\mathrm{tr}A^2)^2/2;\\
&\mathrm{tr}(S^2A^2)' = -2S_1\mathrm{tr}(S^2A^2) - (2\mathrm{tr}S^2 + \mathrm{tr}A^2)\mathrm{tr}(SA^2)\\
&\qquad\qquad -\mathrm{tr}S^3\mathrm{tr}A^2 + S_1\mathrm{tr}S^2\mathrm{tr}A^2 + S_1(\mathrm{tr}A^2)^2/5;\\
&\mathrm{tr}(SAS^2A^2)' = -3S_1\mathrm{tr}(SAS^2A^2).
\end{aligned}
\tag{3}
$$

The seventh equation in (3) can be integrated directly:

$$\mathrm{tr}(SAS^2A^2) = C_7\mathrm{exp}\left[-3\int S_1(t)dt\right],$$

while the first six equations after corresponding simplifications give us the equations:

$$S_1''' + 4S_1S_1''' + 3(S_1')^2 + 6S_1^2S_1' + S_1^4 = 0, \tag{4}$$

$$\gamma''' + 6S_1\gamma'' + 6\gamma'(S_1' + 2S_1^2) + 2\gamma(S_1'' + 3(S_1^2)' + 4S_1^3) - 4S_1\gamma^2/5 = 0. \tag{5}$$

3. It is possible to find the general solutions of (4)-(5) by using the dependence (2). Indeed, considering that $\det(E_3 + tJ(0,\vec{x})) = 1 + c_1t + c_2t^2 + c_3t^3 \equiv q(t) \neq 0, c_1, c_2, c_3,$ being the coefficients of characteristic polynomial of matrix $J(0\vec{x})$, we have found in [5]:

$$J(t,\vec{x}) = q^{-1}(t)\{(1+tc_1)J(0,\vec{x}) - tJ^2(0,\vec{x}) + t^2c_3E_3\},$$

$$S_{ij}(t,\vec{x}) = q^{-1}(t)\{(1+tc_1)S_{ij}(0,\vec{x}) - tS_{ij}^2(0,\vec{x}) - tA_{ij}^2(0,\vec{x}) + t^2c_3\delta_{ij}\},$$

$$A_{ij}(t,\vec{x}) = q^{-1}(t)\{(1+tc_1)A_{ij}(0,\vec{x}) - tA_{ik}(0,\vec{x})S_{kj}(0,\vec{x}) - tS_{ik}(0,\vec{x})A_{kj}(0,\vec{x})\},$$

The time dependence of invariants $S_1(t)$ and $\gamma(t)$ has the form:

$$S_1(t) = \mathrm{tr}S(t,\vec{x}) = q^{-1}(t)(c_1 + 2c_2t + 3c_3t^2) = q'(t)/q(t), \tag{6_1}$$

$$\gamma(t) = -\mathrm{tr}A^2(t,\vec{x}) = q^{-2}(t)(b_0 + b_1t + b_2t^2). \tag{6_2}$$

The formulas ($6_{1,2}$) give general solutions of equations (4)-(5) and contain six constants, presented in terms of space derivatives of the initial (smooth) field of hydrodynamical velocity. But the constants $c_1, c_2, c_3, b_0, b_1, b_2$ can't be chosen arbitrary, being constrained by the equations (4)-(5).

Putting (6_1) in (5), after simplifications, we obtain the equation

$$(q^2\gamma)''' - 4(q^2\gamma)^2 q^{-3} q'/5 = 0.$$

Using here the formula (6_2), we obtain the conditions

$$\gamma(t) = 0 \iff b_0 = b_1 = b_2 = 0 \iff \operatorname{rot} \vec{u}(t, \vec{x}) = 0$$

and/or

$$q'(t) = 0 \iff c_1 = c_2 = c_3 = 0 \iff \operatorname{div} \vec{u}(t, \vec{x}) = 0. \tag{7}$$

Thus we have proved the following important

Theorem 3. *The smooth vector field describing the hydrodynamical velocity of barochronic flow of ideal gas is either potential or solenoid.*

Let us discus each of this two possibilities separately.

A. If the barochronic flow is potential i.e. if

$$\operatorname{rot} \vec{u}(t, \vec{x}) = 0,\ A_{l,k} = (u_{l,k} - u_{k,l})/2 = 0,\ u_{l,k} = u_{k,l} = S_{l,k};\ \vec{u}(t, \vec{x}) = \operatorname{grad} \varphi(t, \vec{x}),$$

then (see [5]) the hydrodynamical velocity has the form (t_0 is an arbitrary constant):

$$\vec{u}(t, \vec{x}) = \vec{x}(t + t_0)^{-1},$$

and the time dependence of density is described by formulas

$$\rho(t) = \rho_0 |1 + t/t_0|^{-3}, \text{ if } |t_0| = -3\rho_0/{\rho_0}' > 0,$$

$$\rho(t) = \rho_1 t^{-3}, \quad \rho_1 > 0, \text{ if } t_0 = 0.$$

B. In the case of solenoid barochronic flow from the formulas ($6_{1,2}$) and (7) we get

$$c_1 = c_2 = c_3 = 0, \quad q(t) = 1, \quad \gamma(t) = b_0 + b_1 t + b_2 t^2.$$

Hence, the matrix $J(0, \vec{x})$ has the rank $r \leq 2$. As it is shown in [5], there is fulfilled the following

Theorem 4. *If the barochron ic flow of ideal gas is solenoid, then all three coefficients of characteristic polynomial of Jacoby matrix are equal to zero for arbitrary moment of time and there are possible the following case:*

1°. $r = 0 \iff \operatorname{tr} S^2 = -\operatorname{tr} A^2 = 0 \iff S = A = 0$;

2°. $r = 1 \iff \operatorname{tr} S^2 = -\operatorname{tr} A^2 > 0,\ \operatorname{tr} S^3 = \operatorname{tr}(SA^2) = 0$ (simple waves)

(then it isevident that $\operatorname{tr}(S^2A^2) = -\operatorname{tr}(S^2)^2/2 < 0,\ \operatorname{tr}(SAS^2A^2) = 0$);

3°. $r = 2$ in all other cases.

$$(r = \operatorname{rank} [u_{i,j}(t, \vec{x})]_{i,j=1,2,3}).$$

Several corollaries follow from the Theorem 4:

Corollary 1. If barochronic flow of ideal gas is solenoid, then the Jacoby matrix $J(t,\vec{x}) = [u_{j,k}(t,\vec{x})]$ is time dependent only and is completely determined up to the real orthogonal transformation by three independent invariants of Jacoby matrix $J(0,\vec{x})$, that do not depend on the space coordinates:

$$\mathrm{tr}\, S^2 = -\mathrm{tr}\, A^2 = b_0, \quad 3\mathrm{tr}\,(SA^2) - \mathrm{tr}\, S^3 = 3b_1/4, \quad \mathrm{tr}(SAS^2A^2),$$

besides this for such flow the invariant $\mathrm{tr}\,(S^2A^2) = (b_2 - b_0^2)/2$ is constrained by the invariants of basis (1) with the inequalities $0 \le b_1^2 \le (2b_0/3)^3$. So, we have:

$$(4\mathrm{tr}(SAS^2A^2))^2 = [\mathrm{tr}S^2(2\mathrm{tr}(S^2A^2) - \mathrm{tr}A^2\mathrm{tr}S^2) - (2\mathrm{tr}(SA^2))^2]^2 - 2(2\mathrm{tr}(S^2A^2) - \mathrm{tr}A^2\mathrm{tr}\, S^2)^3;$$

$$0 \le 6(\mathrm{tr}(S^3)^2 = 54(\mathrm{tr}(SA^2))^2 \le (\mathrm{tr}S^2)^3 = -(\mathrm{tr}A^2)^3 \ge 0. \quad (\mathrm{tr}S = 0)$$

Corollary 2. The hydrodynamical velocity $u_j(t,\vec{x}), (j = 1,2,3)$ of solenoid barochronic flow is described by formulas

$$u_j(t,\vec{x}) = (J_{j,k}(0) - tJ^2_{j,k}(0))(x_k - u^0_k t) + u^0_j, \quad (j = 1,2,3)$$

$$\vec{u}(t,\vec{x}) = (\mathbf{J} - t\mathbf{J}^2)(\vec{x} - \vec{u}^{\,0}t) + \vec{u}^{\,0},$$

where $u^0_j = u_j(0,\vec{0})$, $\langle \vec{e}_j|\mathbf{J}|\vec{e}_k\rangle = J_{jk}(0,\vec{0}) = u_{j,k}(0,\vec{0})$, $(j,k = 1,2,3)$ the initial moment of time and the origin of coordinates being chosen arbitrary according to the condition of barochronity.

Corollary 3. Solution of the simple-wave-type exists iff $\mathrm{tr}S^3 = \mathrm{tr}(SA^2) = 0$. The solutions of the simple-/or double-wave-type can't be merged if the flow remains barochronic, because the corresponding criteria can't be fulfilled simultaneously and do not change in time.

Thus, the invariants (1) of Jacoby matrix allow us to determine the regime of barochronic flow:

a) The flow is potential if $\mathrm{tr}A^2 = (\mathrm{rot}\vec{u})^2 = 0$; then the set of the invariants (1) contains only one independent invariant that is $S_1 = \mathrm{tr}S$ and its initial value determines the initial values and the time dependence of all other invariants, as well as the time dependence of the hydrodynamical velocity $\vec{u}(t,\vec{x})$ and of the gas density $\rho(t)$;

b) The flow is solenoid, if $\mathrm{tr}S = \mathrm{div}\vec{u} = 0$; then there are at most three independent values among invariants (1) (and in the set of their initial values), which completely determine the Jacoby matrix $[u_{j,k}(0,\vec{x})]$ in its (orthonormal) CB [1-4] and together with constants $u^0_j = u_j(0,\vec{0})$ $(j = 1,2,3)$(the initial values of velocity), allow finding the hydrodynamical velocity $\vec{u}(t,\vec{x})$. The six real constants $b_0, b_1, b_2, u^0_1, u^0_2, u^0_3$ are usually independents i.e. in general case their number can't be reduced.

As the three-dimensional barochronic flow physically can be fulfilled only in the infinite homogeneous space that is considered in present paper to be three-dimensional Euclidean space we get the statement:

There exists the nonstationary (potential) solution of Euler equation for hydrodynamical velocity of ideal gas that satisfies (formally) the well known Hubble low in its non relativistic form

$$\vec{u}(t,\vec{r}) = \vec{r}|t + t_0|^{-1} \equiv H\vec{r}. \tag{8}$$

It has to admit that "Hubble constant" H in the equation (8) is $H = |t + t_0|^{-1}$, and therefore the rate of expansion of such "barochronic Universe" has the character of uniform (potential) flow with constant expansion velocity $\vec{u}_0$:

$$\vec{r}(t) = \vec{r}_0 + \vec{u}_0 t.$$

R E F E R E N C E S

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