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ON CARDINALITIES OF AUTOMORPHISM GROUPS OF UNIFORM BINARY RELATION

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Abstract. We give a characterization of all those cardinal numbers, which are representable as the cardinality of the group of all automorphisms of an uniform binary relation. As a consequence, we obtain some statements about the uniformization problems.

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S. Ulam in [1] formulated the following problem:

Problem. Can we find, for every natural number n, a binary relation B on a infinite set E (for example, on a set of cardinality continuum) such that the structure (E, B) has precisely n automorphisms?

A.B. Kharazishvili proved in the theory ZFC that if λ is any infinite cardinal number, and G is a group which $|G| \leq \lambda$, then there exist a set E_G of cardinality λ and binary relation B_G on the set E_G, such that the group of all automorphisms of the structure (E_G, B_G) and the group G are isomorphic (see [2]).

It is easy to see that such a representation theorem implies the positive solution of the Ulams problem. Note that this result of A.B. Kharazishvili also is connected to D. Königs group representation problem for automorphism groups of graphs (see [3]).

In connection with the above-mentioned result the following two natural questions arise:

(a) is it possible to solve Ulams problem effectively (i.e., within ZF theory)?

(b) does there exist a uniform solution of Ulams problem?

Recall that a relation $B \subset E^2$ is uniform binary relation (or a functional relation) if, for any elements $a, b, c \in E$, the following implication holds:

$$(a, b) \in B \ \& \ (a, c) \in B \Rightarrow b = c.$$

Hence, the question (b) establishes communication of Ulam's problem with Luzin's uniformization problem (see [4]).

The works [5] and [6] contain answers to questions (a) and (b). In particular, it is proved that, for every infinite set E and for every positive integer n, there exists a surjective mapping $f : E \to E$ such that the structure (E, f) has precisely n automorphisms.

In other words, Ulam's problem admits the uniform solutions.

So the question can be posed :

(c) whether there exist binary relations which do not admit a uniformization (from the point of view cardinalities of their automorphism groups).

From the aforesaid naturally appears a question on the characterization of all cardinals which are cardinals of automorphism groups of uniform binary relations.

It is obvious that this characterization will be a certain generalization of Ulams problem in one of directions.

Let's note, that in [7] an analogous question is considered for trees of countable height. With the help of that fact, the following theorem is proved.

Theorem 1. *If E is an infinite set, then the following two conditions are equivalent:*

1) there exists a uniform binary relation $f \subset E^2$ such that the cardinality of the group of all automorphisms of the structure (E, f) is equal to λ;

2)$\lambda \in [1, \omega] \cup \{2^\lambda : \lambda \leq |E|\}$.

From Theorem 1 and the above-mentioned result of A.B. Kharazishvili, it can be deduced the next statement.

Theorem 2. *There exists an infinite set E and a binary relation $S \subset E^2$ such that, for every uniform binary relation $f \subset E^2$, the cardinalities of the automorphisms groups of the structures (E, S) and (E, f) are distinct.*

Obviously, this is the answer to question (c). Let us point out the following three consequences of Theorem 1.

Corollary 1. It is impossible to represent all infinite groups as automorphism groups of uniform binary relations.

Corollary 2. Let λ be a cardinal such that there exists a binary relation on the set of all natural numbers which has precisely λ automorphisms and does not exist a uniform binary relation which has precisely λ automorphisms. Then $\omega < \lambda < 2^\omega$;

Corollary 3. The proposition:

"There exists a uniform binary relation which has precisely ω_1 automorphisms" is equivalent to the continuum hypothesis.

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