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ON THE EXISTENCE OF SOLUTION TO NON-LOCAL BOUNDARY PROBLEM

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Abstract. One non-local boundary value problem for sixth order partial differential equation is considered.

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The non-local boundary problem has been considered in [1-8]. In the case of smooth domains the non-local boundary problem for the Laplace equation in the space R^3 is posed as follows: Let Ω be a simply connected bounded domain from the class $C^{(2,\alpha)}$, $0<\alpha\leq 1$, S be a closed surface from $C^{(2,\alpha)}$ belonging to Ω ($S\subset\Omega$), and let $\zeta=z(x)$ be a $C^{(2,\alpha)}$-diffeomorphism from Ω onto S. Assume that the boundary function $f\in C(\partial\Omega)$. One has fined a function $\varphi\in C(\partial\Omega)$ satisfying the boundary condition

$$\varphi-K\varphi=f,$$

where

$$K\varphi(x)=v(z(x))=-\int\limits_{\partial\Omega}\frac{\partial G(z(x),y)}{\partial\nu_y}\varphi(y)dS_y.$$

For a biharmonic equation the non-local boundary problem in a circle is considered in the monograph [9, p. 312]. In this paper we consider the non-local boundary problem for the equation $\Delta^3 v=0$ in a smooth domain Ω.

Let Ω be a simply connected bounded domain from $C^{(5,\alpha)}$, $0<\alpha\leq 1$, S be a smooth closed surface from $C^{(5,\alpha)}$ ($S\subset\Omega$). Let further $\zeta=z(x)$ be a diffeomorphism from $\partial\Omega$ onto S, $z(x)\in C^{(5,\alpha)}$, ℓ_x be a smooth direction at a point $x\in\partial\Omega$, $\ell_x\in C^{(4,\alpha)}$ for which $\cos(\nu_x\hat{}\ell_x)\neq 0$, $x\in\partial\Omega$, ν_x be the outer normal.

Let us fined a solution to the equation $\Delta^3 v=0$ in Ω from $C^{(5,\alpha')}(\overline{\Omega})$, $0<\alpha'<\alpha\leq 1$ satisfying the following boundary conditions:

$$\begin{aligned} v(x)-v(z(x))&=f(x), && f\in C^{(5,\alpha)},\\ \frac{\partial v(x)}{\partial\ell_x}&=g_1(x), && g\in C^{(4,\alpha)},\\ \frac{\partial^2 v(x)}{\partial\ell_x^2}&=g_2(x), && g_2\in C^{(3,\alpha)}. \end{aligned}\tag{1}$$

If v is a solution to the problem (1) belonging to $C^{(5,\alpha')}(\overline{\Omega})$, then it takes place the following representation

$$v(x)=H_0(x)-\int\limits_{\Omega}G(x,y)H_1(y)dy+\int\limits_{\Omega}G(x,y)\int\limits_{\Omega}G(y,z)H_2(z)dzdy,\tag{2}$$

where H_0, H_1, H_3 are harmonic functions in Ω satisfying the following boundary conditions: $H_0(x) = v(x) = \varphi(x)$, $H_1(x) = \Delta v(x)$, $H_2(x) = \Delta^2 v(x)$, $x \in \partial\Omega$, G is the Green function for the Dirichlet problem. For the Green function we have [10]

$$\frac{\partial G(x,y)}{\partial \ell_y} = \frac{\partial G(x,y)}{\partial \nu_y} \cos(\nu_y \hat{} \ell_y), \quad y \in \partial\Omega,\ x \in \Omega. \tag{3}$$

Besides, using equalities from [11, p. 115] one can prove that

$$\frac{\partial^2 V_i^F(x)}{\partial \ell_x^2} - \frac{\partial^2 V_e^F(x)}{\partial \ell_x^2} = 4\pi F(x) \cos^2(\nu_x \hat{} \ell_x), \quad F \in C^2(\overline{\Omega}),$$

where

$$V^F(x) = \int\limits_{\Omega} \frac{F(y)dy}{|x-y|}, \quad U^{Tf}(x) = \int\limits_{\partial\Omega} \frac{f'(y)dS_y}{|x-y|}, \quad Tf(y) = f'(y) = -\int\limits_{\Omega} \frac{\partial G(x,y)}{\partial \nu_y} f(x)dx,$$

and V_i^F, V_e^F denote respectively the inside limit and outside limit.

Formula (2) implies that

$$v(x) = H_0(x) = \int\limits_{\Omega} G(x,y)F(y)dY, \quad F(y) = H_1(y) - \int\limits_{\Omega} G(y,z)H_2(z)dz.$$

Using this equalities and the second boundary condition of (1) we get

$$\frac{\partial v(x)}{\partial \ell_x} = g_1(x) = \frac{\partial H_0(x)}{\partial \ell_x} - \int\limits_{\Omega} \frac{\partial G(x,y)}{\partial \ell_x} F(y)dy \quad x \in \partial\Omega.$$

This and (3) imply

$$g_1(x) = \frac{\partial H_0(x)}{\partial \ell_x} - F'(x)\cos(\nu_x \hat{} \ell_x), \quad F'(x) = \frac{1}{\cos(\nu_x \hat{} \ell_x)} \left[\frac{\partial H_0(x)}{\partial \ell_x} - g_1(x) \right].$$

By virtue of the boundary condition of (1) we have

$$\frac{\partial^2 v}{\partial \ell_x^2} = \frac{\partial^2 H_0(x)}{\partial \ell_x^2} - \frac{\partial^2}{\partial \ell_x^2}\big[V^F(x) - U^{F'}(x)\big] = \frac{\partial^2 H_0(x)}{\partial \ell_x^2} - \frac{\partial^2 V_i^F(x)}{\partial \ell_x^2} + \frac{\partial^2 U_i^{\Psi_1}(x)}{\partial \ell_x^2}$$
$$= \frac{\partial^2 H_0(x)}{\partial \ell_x^2} - \left[\frac{\partial^2 V_i^F(x)}{\partial \ell_x^2} - \frac{\partial^2 V_\ell^F(x)}{\partial \ell_x^2} \right] + \frac{\partial^2 U_i^{\Psi_1}(x)}{\partial \ell_x^2} - \frac{\partial^2 U_\ell^{\Psi_1}(x)}{\partial \ell_x^2} = \frac{\partial^2 H_0(x)}{\partial \ell_x^2}$$
$$-4\pi F(x)\cos^2(\nu_x \hat{} \ell_x) + \frac{\partial^2 U_i^{\Psi_1}(x)}{\partial \ell_x^2} - \frac{\partial^2 U_\ell^{\Psi_1}(x)}{\partial \ell_x^2}, \quad \Psi_1 = F' = TF.$$

Let us use again the second boundary condition for representation (2). We get

$$\frac{\partial v}{\partial \ell_x} = g_1(x) = \frac{\partial H_0(x)}{\partial \ell_x} + H_1'(x) - \big(V_G^{H_2}\big)'(x)\cos(\nu_x \hat{} \ell_x) \quad x \in \partial\Omega. \tag{4}$$

Here

$$V_G^{H_2}(x) = \int\limits_{\Omega} G(x,y) H_2(y) dy, \quad \big(V_G^{H_2}\big)'(x) = TV_G^{H_2}(x). \tag{5}$$

Let us define $TV_G^{H_2}\big(TV_G^{H_2} = \Psi_2\big)$ from (5)

$$TV_G^{H_2}(x) = \Psi_2(x) = \frac{1}{\cos(\nu_x \hat{} \ell_x)} \left[\frac{\partial H_0(x)}{\partial \ell_x} + H_1'(x) - g_1(x)\right] \in C^{(4,\alpha_1)}. \tag{6}$$

For representation (2) let us apply the first boundary condition of problem (1)

$$v(x) - v(z(x)) + \int\limits_{\Omega} G(z(x),y) H_1(y) dS_y - \int\limits_{\Omega} G(z(x),y) V_G^{H_2}(y) dS_y = f(x).$$

According to the definition of the operator T we have

$$\int\limits_{\partial\Omega} \frac{V_G^{H_2}(y)dy}{|x-y|} = \int\limits_{\partial\Omega} \frac{\Psi_2(y) dS_y}{|x-y|}, \quad x \in R^3 - \Omega, \quad \left(v_1(x) = \int\limits_{\Omega} \frac{V_G^{H_2}(y)dy}{|x-y|}, \ x \in \overline{\Omega} \right).$$

In order to find H_2 let us pose the Dirichtet problem for the equation $\Delta^3 v_1(x) = 0$ $(\Psi_2 \in C^{(4,\alpha_1)}, \ U^{\Psi_2} \in C^{(5,\alpha_1')}_{(\overline{\Omega})}, \ \alpha_1' < \alpha_1 < \alpha)$

$$\begin{aligned} v_1(x) &= U^{\Psi_2}(x), && x \in \partial\Omega, \\ \frac{\partial v_1(x)}{\partial \nu_x} &= \frac{\partial U_\ell^{\Psi_2}(x)}{\partial \nu_x}, && x \in \partial\Omega, \\ \frac{\partial^2 v_1(x)}{\partial \nu_x^2} &= \frac{\partial^2 U_\ell^{\Psi_2}(x)}{\partial \nu_x^2}, && x \in \partial\Omega. \end{aligned}$$

It is well-known that there exists a solution v_1 to the Dirichlet problem belonging to the class $C^{(5,\beta)}(\overline{\Omega})$, $0 < \alpha' < \beta < \alpha_1' < \alpha_1 < \alpha$. The equality $\Delta^2 v_1(x) = H_2(x)$ implies that $H_2 \in C^{(1,\beta)}(\overline{\Omega})$. From here and (4) by virtue of (6) we get the second kind Fredholm integral equation

$$\varphi - K\varphi = g, \tag{7}$$

where K is a compact operator from the space $C^{(5,\alpha')}$ into the space $C^{(5,\alpha')}$; the right-hand-side belongs $C^{(5,\alpha')}$ and depends on $g_1, \ g_2$ and f only.

It is not difficult to make sure that the null-space of the operator A, $A\varphi = \varphi - K\varphi$ is one-dimensional. By virtue of the Reisz-Shauder theory, the null-space of the operator $A^* = I^* - K^*$ mapping $\{C^{(5,\alpha')}\}^*$ into itself is also one-dimensional. Therefore, there exists a unique functional $\Phi_1 \in \{C^{(5,\alpha')}\}^*$ which is an eigenvalue of a dual compact operator K^*, i.e. $K^*\Phi_1 = \Phi_1$. Let us define the following space with elements from $C^{(5,\alpha')}(\partial\Omega)$

$$B_1 = \{g : g \in C^{(5,\alpha')}, \ \Phi_1(g) = 0\}.$$

According to the Reisz-Shauder theory, the equation (7) is solvable if and only if $g \in B_1$. This implies the solvability of problem 1, if the right-hand-side of (7) belongs to B_1.

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